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PLANAR LIBRATIONS OF A GRAVITY-ORIENTED SATELLITE UNDER THE INFLUENCE OF SOLAR RADIATION PRESSURE†

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Abstract—Satellite oscillations about its centre of mass in the circular orbit plane are dealt with. The satellite is assumed symmetrical about a plane permanently coinciding with the orbit plane. A gravity-gradient torque and a torque of solar radiation pressure on an unshadowed flat plate—a part of the satellite—are taken into account. The centre of pressure is supposed to belong to the principal axis of inertia. Effects of entering the Earth's shadow are neglected. A simplification that the orbit lies in the ecliptic plane is adopted. Under the assumptions made, the satellite motion is described by a non-autonomous differential second-order equation. A problem is to find symmetrical and nonsymmetrical periodic motions of orbital period and to determine their stability. For the case of small radiation disturbance, the Krylov–Bogolyubov asymptotic approach is used in the analysis. The libration in the vicinity of the main resonance has been elaborated. For the satellite dynamically resembling a sphere the investigation is treated with the Volosov–Morgunov averaging method. A resonant value of the radiation torque parameter has been found. A question of periodic motions bifurcation is cleared up. For the satellite with an arbitrary tensor of inertia under non-small radiation disturbance the problem has been solved numerically. The main results are represented as a chart graphically demonstrating regions of existence and stability of possible periodic librations of the satellite on parametric plane.

1. INTRODUCTION

A possible way to the passive stabilization of artificial satellites is to use the gravity gradient of a planet, due to which the torques arise and turn a satellite. It seems expedient to use such a system of passive stabilization also for spacecraft or objects that have a rather large area of surface reflecting and absorbing the solar radiation. For example, these are communication and meteorological satellites, transmitting antennae of future solar energy satellites, various space platforms, etc. However, if the orbit is high enough and a satellite is large, one should take into account the torques arising due to the solar radiation pressure onto different parts of the satellite. This factor may result in the attitude instability of the gravity-oriented satellite, though it may also be used for passive stabilization of the satellite[1].

In this paper for the simplest model of a symmetric satellite in a circular orbit which undergoes an effect of the gravity and light torques, an attempt was made to study the librational motion and stability of periodic oscillations as functions of the basic parameters. A similar problem was solved in[2], where an effect of the light torque upon the gravity-oriented symmetric plate in an elliptic orbit was studied. In this paper a different model is taken to imitate the umbrella-like satellites whose main reflecting surface in the regime of gravitational orientation is approximately orthogonal to the line directing to the gravity centre. For this model the zones of parametric instability of oscillations in the satellite's orbit plane are established when the gravity torque stabilization regime is violated due to the effect of the solar radiation pressure torque. Appearance of such an instability should be taken into consideration in designs of spacecraft that possess the above properties as well as in designs of the gravity torque stabilization systems.

2. MODEL AND THE MOTION EQUATIONS

Let us consider a symmetric satellite in a circular orbit (Fig. 1). For simplicity we assume that the orbit lies in the ecliptic plane. We restrict ourselves to studying the motion of the satellite about its centre of mass in the orbital plane. Let us take into consideration an effect of the gravity torque caused by a gradient of the planet's central gravity field as well as an effect of the solar radiation pressure torque.

In Fig. 1, the satellite's centre of mass is designated by O, and its central major axes of inertia by Ox and Oy with the inertia moments A and B, respectively. The third axis of inertia, Oz with the moment C is perpendicular to the orbital plane (Fig. 1). An angle between the radius vector EO of the satellite and the axis Ox is designated by δ.

We shall take into account an effect of the light pressure only upon a part of the satellite—the opaque

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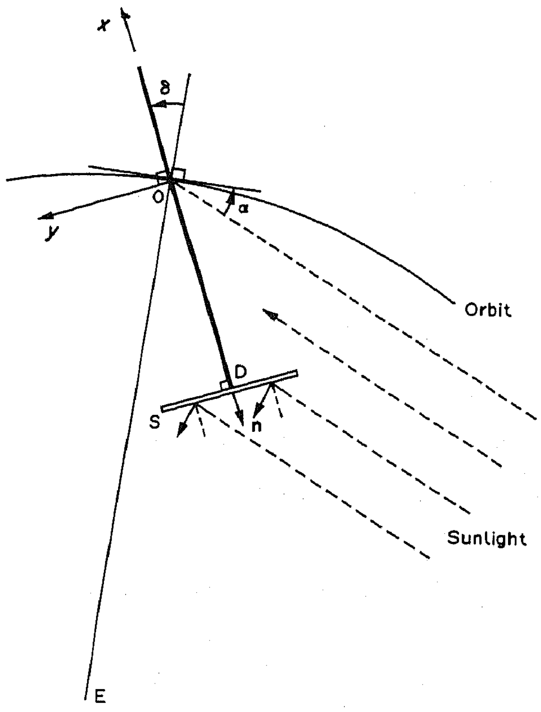

Fig. 1. Symmetric satellite in a circular orbit.

plate S the normal to which is collinear to the axis Ox. It is assumed that the surface area of other parts is negligibly small. Effects of mutual obscuration and entering the Earth's shadow are ignored. The latter can be justified because a relative value of the light torque becomes significant in high orbits where the shadowing time is small. The plate S is assumed to be symmetric with respect to the plane Oxy, the pressure centre D being on the axis Ox.

Using the formulas and results from[3] the equation of the satellite oscillations can be written in the form

$$\frac{d^2\phi}{d\alpha^2} + 4\omega^2 \sin\phi + 8h\left|\sin\left(\frac{\phi}{2}+\alpha\right)\right| \times \cos\left(\frac{\phi}{2}+\alpha\right) = 0. \quad (1)$$

Here $\phi = 2\delta$; α is the angle between the direction to the Sun and a tangent to the satellite orbit; $\omega^2 = 3(B-A)\omega_0^2/4C\omega_r^2$ is the parameter that determines the gravitational torque (ω_0 is an absolute angular velocity of the satellite's centre-of-mass motion along the orbit, ω_r is the angular velocity of the satellite's centre of mass with respect to the Sun); $h = p_c S(1-\epsilon_c)r/4C\omega_r^2$ is the parameter that determines the light torque [p_c is a constant of the solar radiation pressure, in the Earth's orbit $p_c = 4.64 \times 10^{-6}$ Pa, $\epsilon_c (0 \leqslant \epsilon_c \leqslant 1)$ is the reflection coefficient which is the same on both sides of the plate surface, the area of each side is designated by S]; $r = |OD|$ is a distance between the centre of mass and the centre of pressure. A mirror law of reflection is assumed.

In eqn (1) the term corresponding to the light torque is proportional to the absorption coefficient equal to $(1-\epsilon_c)$. It can be explained by the accepted properties of the satellite's symmetry.

The model described above approximates a rather wide class of physical objects, in particular, a satellite consisting of an antenna disk and an orthogonal central bar.

In the absence of the light torque ($h = 0$) and with $\omega^2 > 0$ eqn (1) allows the stable solution $\delta = 0$, it is the so-called gravity torque stabilization regime[3]. For small $h \neq 0$ the satellite will generally perform small oscillations about its position. These oscillations will be stable for all possible values $\omega^2 > 0$ and small h except for those which get into regions of parametric resonance. In the linear approximation the zones boundaries of parametric resonance for the trivial solution of eqn (1) coincide with such for the equation

$$\frac{d^2\phi}{d\alpha^2} + 4[\omega^2 + h(\operatorname{sgn}\cos\alpha)\cos 2\alpha]\,\phi = 0, \quad (2)$$

which is called the Mathieu–Meissner equation since it represents a combination of the Mathieu equation (the harmonic excitation) and the Meissner equation (excitation of meander type)[4].

Boundaries of the parametric resonance zones in the plane of parameters ω^2, h for eqn (2) are determined from the condition $|trA^*| = 2$[5], where A^* is the matrix of monodromy of eqn (2) the period of which is 2π. In order to determine approximately the matrix of monodromy approximate solutions of eqn (2) should be constructed. It can be done by using an asymptotic Krylov–Bogolyubov method[6]. Expressions for the boundaries of the first two instability zones in the linear approximation are:

$$\omega^2 \approx \frac{1}{16} \pm \frac{2h}{3\pi}, \quad \omega^2 \approx \frac{9}{16} \pm \frac{6h}{5\pi}. \quad (3)$$

3. FORCED OSCILLATIONS

We studied small oscillations of the satellite in the vicinity of main resonance, i.e. for ω^2 being close to $1/4$. We shall assume that oscillations are near harmonic, and write eqn (1) in the corresponding form:

$$\frac{d^2\phi}{d\alpha^2} + 4\omega^2\phi = -8h\left|\sin\left(\frac{\phi}{2}+\alpha\right)\right| \times \cos\left(\frac{\phi}{2}+\alpha\right) + 4\omega^2(\phi - \sin\phi) \equiv f(\alpha, \phi). \quad (4)$$

By assuming that f is small and following the Krylov–Bogolyubov method[6] we shall search for the solution of eqn (4) in the vicinity of main resonance as the first approximation in the form:

$$\phi = 2a\cos\psi, \quad \psi = \alpha + v.$$

The amplitude a and the phase v are determined from the system of equations:

$$\frac{da}{d\alpha} = A_1(a, v),$$

$$\frac{dv}{d\alpha} = 2\omega - 1 + B_1(a, v), \tag{5}$$

where A_1 and B_1 are particular periodic-in-v solutions of the system

$$\tfrac{1}{2}(2\omega - 1)\frac{\partial A_1}{\partial v} - 4\omega a B_1 = \frac{1}{2\pi}\int_0^{2\pi} f_0 \cos\psi \, d\psi,$$

$$(2\omega - 1)\, a\frac{\partial B_1}{\partial v} + 2\omega A_1 = -\frac{1}{2\pi}\int_0^{2\pi} f_o \sin\psi \, d\psi,$$

$$f_0 = f(\psi - v, 2a\cos\psi). \tag{6}$$

Developing f_0 as a Fourier series we calculate integrals in the right-hand side of (6). In the expressions for them we shall keep only harmonics not higher than the first. After that instead of eqn (6) we obtain the system

$$\tfrac{1}{2}(2\omega - 1)\frac{\partial A_1}{\partial v} - 4\omega a B_1 = -\frac{16h}{3\pi}[J_o(a) - J_2(a)]\cos v + 4\omega^2[a - J_1(2a)],$$

$$(2\omega - 1)\, a\frac{\partial B_1}{\partial v} + 2\omega A_1 = \frac{16h}{3\pi}[J_0(a) + J_2(a)]\sin v. \tag{7}$$

Here $J_k(x)$ is the kth order Bessel function of the first kind.

Obtaining the periodic-in-v expressions for A_1 and B_1 from (7) we shall give the system (5) in the form:

$$\frac{da}{d\alpha} = \frac{16h}{3\pi}\frac{[J_0(a)(6\omega - 1) + J_2(a)(2\omega + 1)]}{(6\omega - 1)(2\omega + 1)} \times \sin v \equiv -\frac{1}{k}\frac{\partial\Phi}{\partial v},$$

$$\frac{dv}{d\alpha} = \omega - 1 + \frac{\omega}{a}J_1(2a) + \frac{16h}{3\pi} \times \frac{[J_0(a)(6\omega - 1) - J_2(a)2\omega + 1)]}{a(6\omega - 1)(2\omega + 1)} \times \cos v \equiv \frac{1}{k}\frac{\partial\Phi}{\partial a}. \tag{8}$$

Omitting the calculations we shall give the expressions for the integrating factor k and the function Φ with an accuracy to a^4:

$$\Phi = 2a\left[\frac{32h}{3\pi}\left((6\omega - 1) - (2\omega + 1)\frac{a^2}{8}\right) \times \cos v + (2\omega + 1) \times a\left((6\omega - 1)(2\omega - 1) - (2\omega^2 + 3\omega - 1)\frac{a^2}{4}\right)\right],$$

$$k = (2\omega + 1)a[2(6\omega - 1) + (2\omega - 1)a^2],$$

$$\Phi(a, v, \omega, h) = \Phi(-a, v + \pi, \omega, h),$$

$$k(a, \omega) = -k(-a, \omega). \tag{9}$$

Equations (8) have the primitive integral $\Phi = \Phi_0 =$ const. Figures 2 and 3 show the integral curves in the plane a, v for $h = 0.05$, $\omega = 0.51$ and $\omega = 0.60$, respectively. For the case in Fig. 2 there is one stable stationary periodic regime for $v = 0$, for the case in Fig. 3 there are additionally stable and unstable stationary regimes for $v = \pi$.

Let us dwell on the stationary periodic regimes. But first let us note that in a sufficiently wide neighbourhood of the resonant point $\omega = 0.5$ and for not very large values of amplitude $a \neq 0$, the factor k does not vanish. Stationary regimes of motion correspond to critical points of function Φ, which are given by the system $\partial\Phi/\partial v = 0$, $\partial\Phi/\partial a = 0$. In order to satisfy the first of these equations it is sufficient to put $v = \pi n$, $n \in Z$. By using the symmetry property of Φ we restrict ourselves to consideration of the case $v = 0$. The critical amplitude is determined by the equation

$$\frac{\partial\Phi(a, 0, \omega, h)}{\partial a} = 0. \tag{10}$$

The form of critical manifold M (10) in the space a, ω^2, h in the vicinity of singular critical point $a = 0$ is shown in Fig. 4. Two parameters ω and h provide a universality of deformation of function Φ in the vicinity of the cusp point $a = 0$[7]. This singularity of Φ is structurally stable, hence it is justified to reject terms of the order a^5 and higher in expansion (5).

The curve S in Fig. 4 is a singular set (where $\partial^2\Phi/\partial a^2 = 0$), whose projection onto the plane of parameters ω^2, h forms a bifurcation curve. It is constructed in Fig. 5 and marked by B′. Here, for comparison, the bifurcation curve B obtained by numerical integration of the original equation is given [see (3)]. As it is seen, curve B′ rather well approximates the curve B even for relatively large h, which proves an adequacy of the approximate method. To the left of curve B′(B) (Fig. 5) there is one 2π-periodic solution of eqn (1) (cf. with Fig. 2), and to the right—three such solutions (cf. with Fig. 3).

Crossing the critical manifold M by the planes $h =$ const gives a set of amplitude–frequency responses

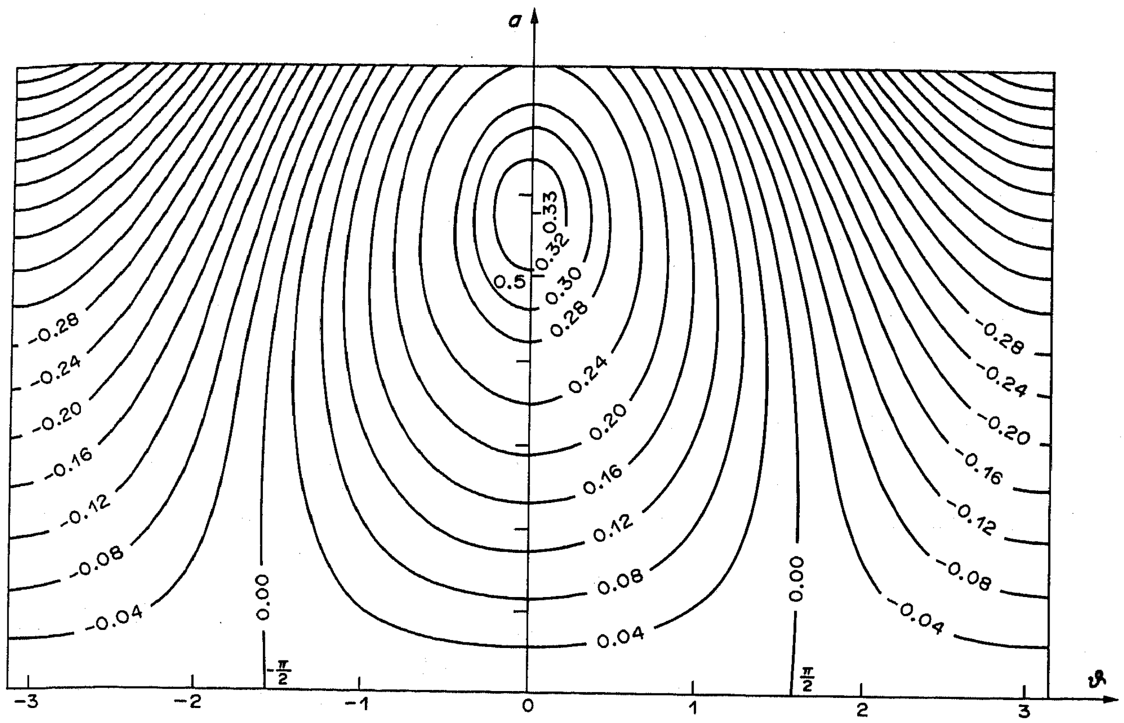

Fig. 2. Integral curves at $\omega^2 = 0.26$, $h = 0.05$.

some of which are given in Fig. 6 for h varying from 0 to 0.025. The dashed lines correspond to unstable regimes.

4. PERIODIC SOLUTIONS

By introducing a new independent variable $t = \alpha + \pi/2$ we write eqn (1) in the form

$$\frac{d^2\phi}{dt^2} + 4\omega^2 \sin\phi + 8h\left|\cos\left(\frac{\phi}{2} + t\right)\right| \times \sin\left(\frac{\phi}{2} + t\right) = 0. \quad (11)$$

For eqn (11) whose period is 2π we shall put the boundary value problem of searching for periodic solutions with the same period:

$$\phi(0) = \phi(2\pi), \quad \frac{d\phi}{dt}(0) = \frac{d\phi}{dt}(2\pi). \quad (12)$$

Additionally, if we require that the periodic solutions be odd, the boundary value problem (12) will become equivalent to:

$$\phi(0) = \phi(\pi) = 0. \quad (13)$$

It follows from the fact that eqn (11) is invariant with respect to the simultaneous change of signs at t, ϕ and $d^2\phi/dt^2$.

If $h = 0$, the solution of problem (11) and (13) exists for any ω^2[8]. As it is noted in[9], in this case only one solution, $\phi_1 = 0$, is allowed by (11) and (13) for $-\frac{3}{4} \leqslant \omega^2 < \frac{1}{4}$, and there are three solutions for $\frac{1}{4} \leqslant \omega^2 \leqslant \frac{3}{4}$: $\phi_1 = -2\arcsin(k \cdot sn\, 2\omega t)$, $\phi_2 = 2\arcsin(k \cdot sn\, 2\omega t)$, $\phi_3 = 0$, where $k^2 = 1/16\omega^2 (d\phi/dt)^2_{t=0}$. These solutions can be considered as generating for eqn (13) if $h \neq 0$.

The solutions of the boundary value problem (11) and (13) are plotted in Figs 7 and 8 for two pairs of values of ω^2, h. These solutions are obtained numerically. Figure 7 corresponds to $\omega^2 = -0.048$, $h = 0.964$, while Fig. 8 to $\omega^2 = 0.063$, $h = 0.011$. Below it can be seen from the stability chart that the first pair of values refers to a stable solution, while the second pair is located in the region of parametric instability.

It seems interesting to find the regions of stable and unstable periodic solutions generated by ϕ_1, ϕ_2 and ϕ_3 in the plane of determining parameters ω^2, h, and not only for small h. It is possible by using the following numerical algorithm. First, for a certain pair of ω^2, h we solve problem (11) and (13). Then, integrating numerically the equation in variations for (11) on an interval $[0, 2\pi]$ we obtain the monodromy matrix $\tilde{A}$ for this equation. After this, the function $R(\omega^2, h) \equiv |tr\tilde{A}| - 2$ can be calculated. By using the Newton method we determine ω^2, h such that $R(\omega^2, h) = 0$, i.e. the values belonging to the boundary of stability zone[5]. Further, we construct the boundary curve by applying some step. The periodic solution is stable in the first approximation if $R(\omega^2, h) < 0$, and it is unstable if $R(\omega^2, h) > 0$. The necessary conditions for stability of periodic solutions of (11) that are obtained in the first approximation case, are also sufficient for nearly all ω^2, h.

Let us consider the stability chart for solutions ϕ_1 and ϕ_3. The solution ϕ_1 exists in the entire region of varying ω^2 and at least for h given in Fig. 9. The solutions ϕ_2 and ϕ_3 exist for the ω^2 that exceed the ones on the bifurcation curve B, and the solution ϕ_2 is unstable (cf. Fig. 6).

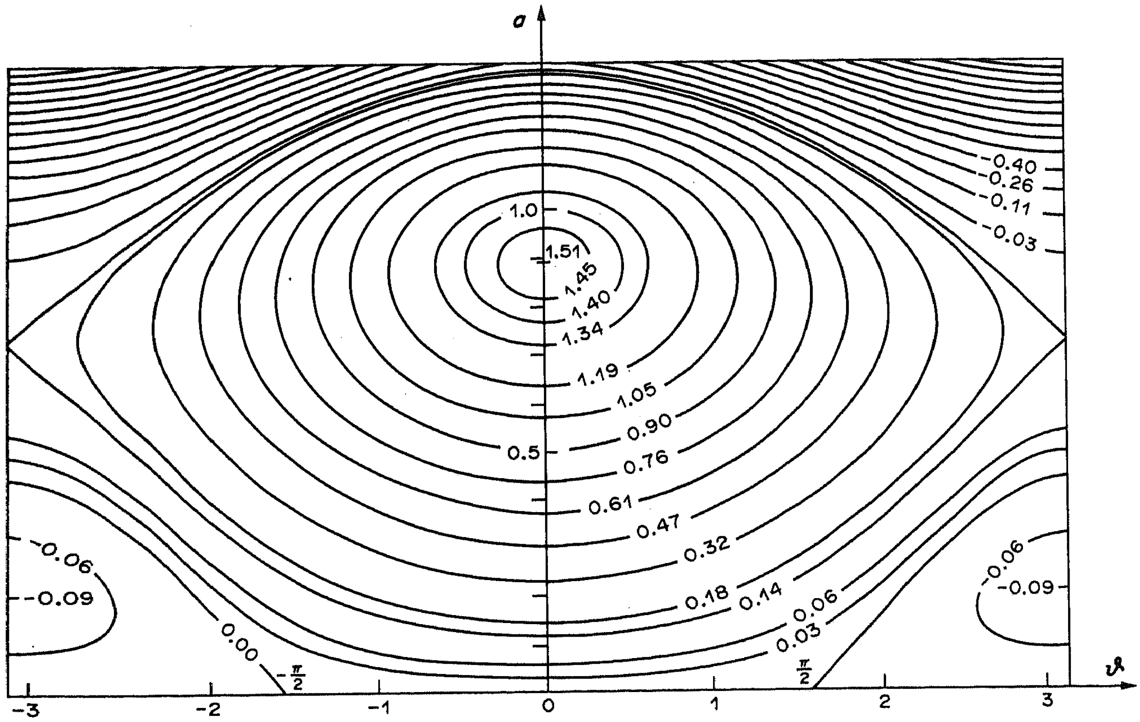

Fig. 3. Integral curves at $\omega^2 = 0.36$, $h = 0.05$.

For the given orbit, the values of ω^2 that are physically feasible lie in the band $|\omega^2| \leqslant 3\omega_0^2/4\omega_r^2$. It follows from validity of the inequality:

$$I_1 + I_2 \geqslant I_3 \quad (I_1, I_2, I_3 \in \{A, B, C\}).$$

For overwhelming majority of existing and future satellites $\omega_0 \approx \omega_r$, so a range of different ω^2 varying up to 1 in Fig. 9 seems to be quite sufficient from the viewpoint of practical applications.

The first zone of parametric resonance (solution ϕ_1) begins at the point $\omega^2 = \frac{1}{16}$, $h = 0$; in the linear approximation its boundaries are described by the first formula of (3). It is interesting to note the transition of stability zone into the region of negative ω^2.

The second zone of parametric instability (solution ϕ_3) begins at the point $\omega^2 = \frac{9}{16}$, $h = 0$. For large h its boundaries intersect, which is typical for the stability diagram of the Meissner equation. In the linear approximation for small h the zone boundaries are described by the second formula of (3).

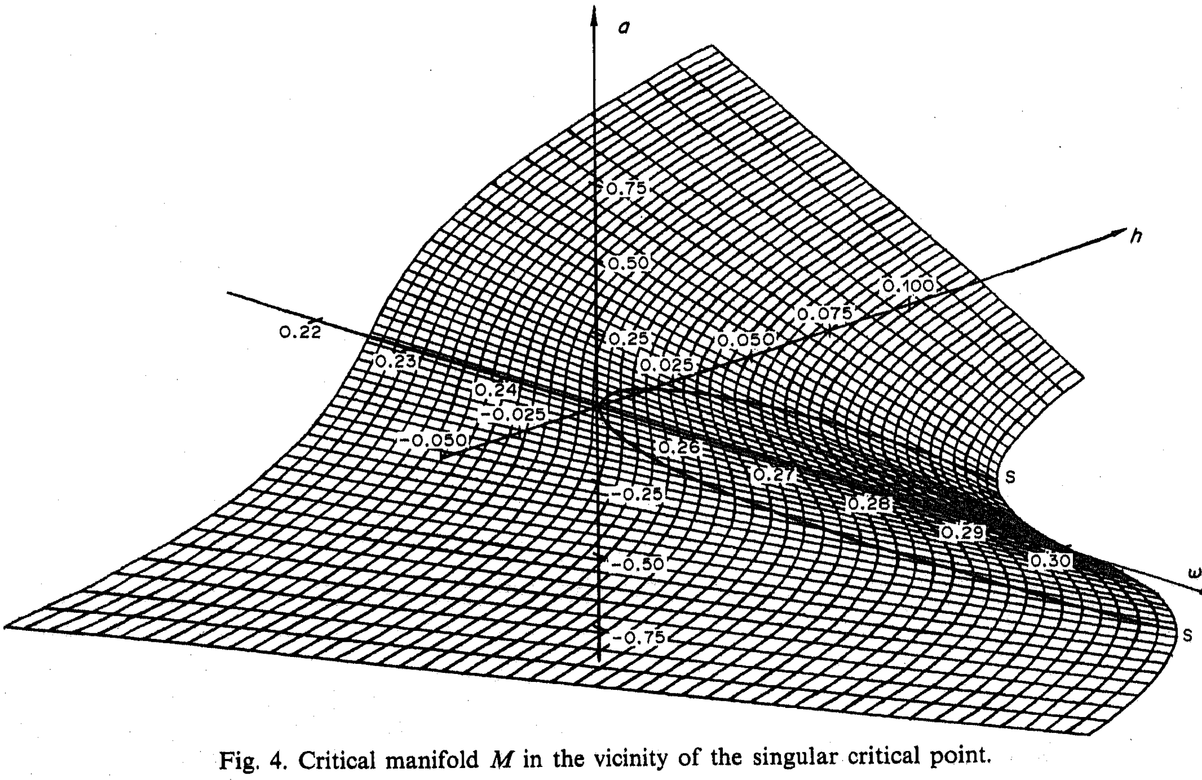

Fig. 4. Critical manifold M in the vicinity of the singular critical point.

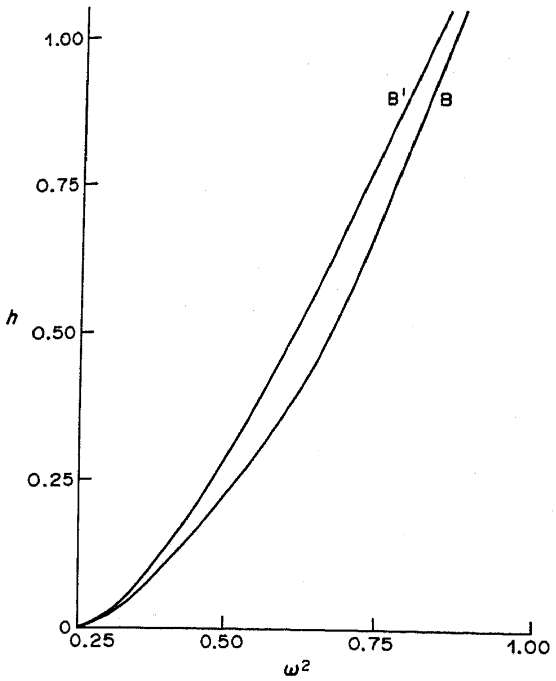

Fig. 5. Approximate bifurcation curve B' in comparison with bifurcation curve B obtained by numerical integration.

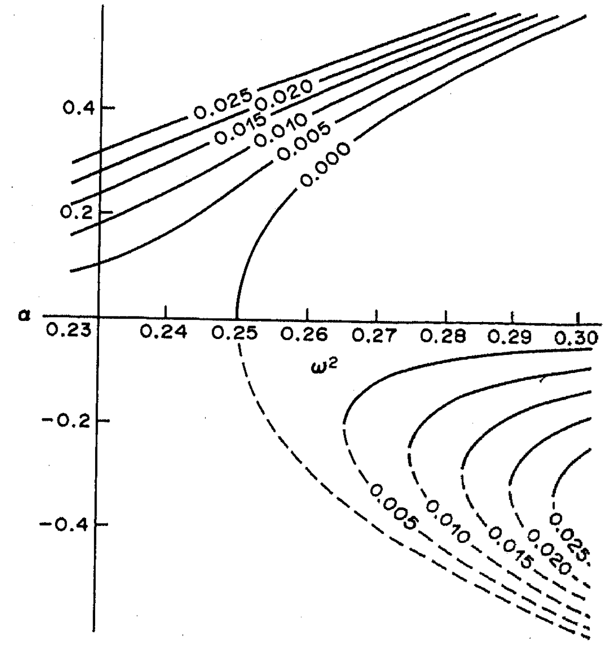

Fig. 6. Amplitude-frequency responses for light parameter h varying from 0 to 0.025. The dashed lines correspond to unstable regimes.

The straight line $h = \omega^2$ has a sense of equality between maximal values of gravitational and light torques. It is interesting to note that the intersection points of the second zone boundaries lie nearly on the straight lines $h = \omega^2/2$ and $h = \omega^2$, and the straight line $h = \omega^2$ is "tangent" to the boundary of instability zone of solution ϕ_1.

Calculation of the bifurcation curve B was carried out by using the numerical algorithm different from that described above. It is as follows. First, for some ϕ' the values of ω^2, h are obtained by the Newton method such that $\phi(0) = 0$, $d\phi/dt(0) = \phi'$ determine the odd 2π-periodic solution of (11) with these ω^2, h. The monodromy matrix A^* is calculated for the equation in variations. After this by using the Newton method the zero of function is $S(\phi') \equiv |\mathrm{tr}\tilde{A}| - 2$ is searched for. When it is found the corresponding values of ω^2, h determine the point of the bifurcation curve. Finally, the construction of this curve is performed with a certain step. This algorithm can be used

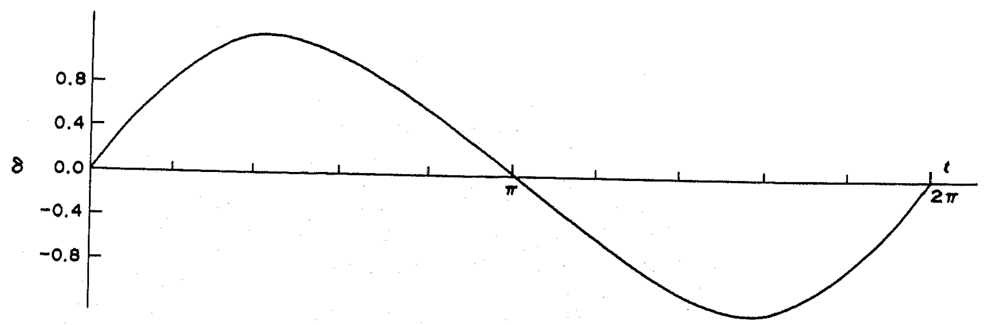

Fig. 7. The stable odd solution of the boundary value problem (11) and (13) at $\omega^2 = -0.048$, $h = 0.964$.

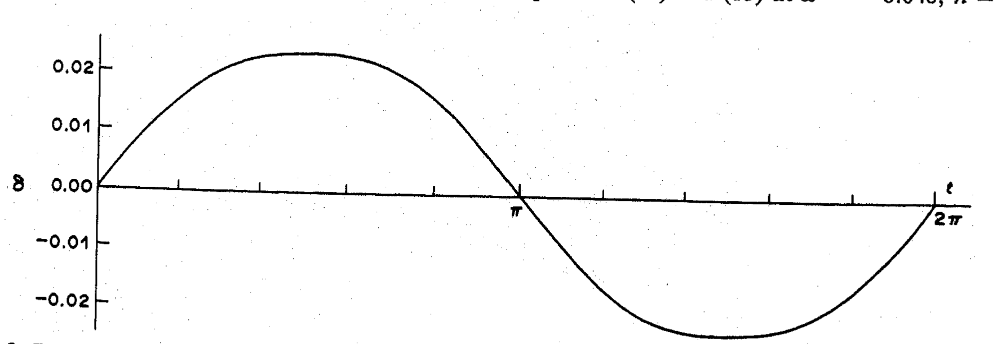

Fig. 8. The unstable odd solution of the boundary value problem (11) and (13) at $\omega^2 = 0.063$, $h = 0.011$.

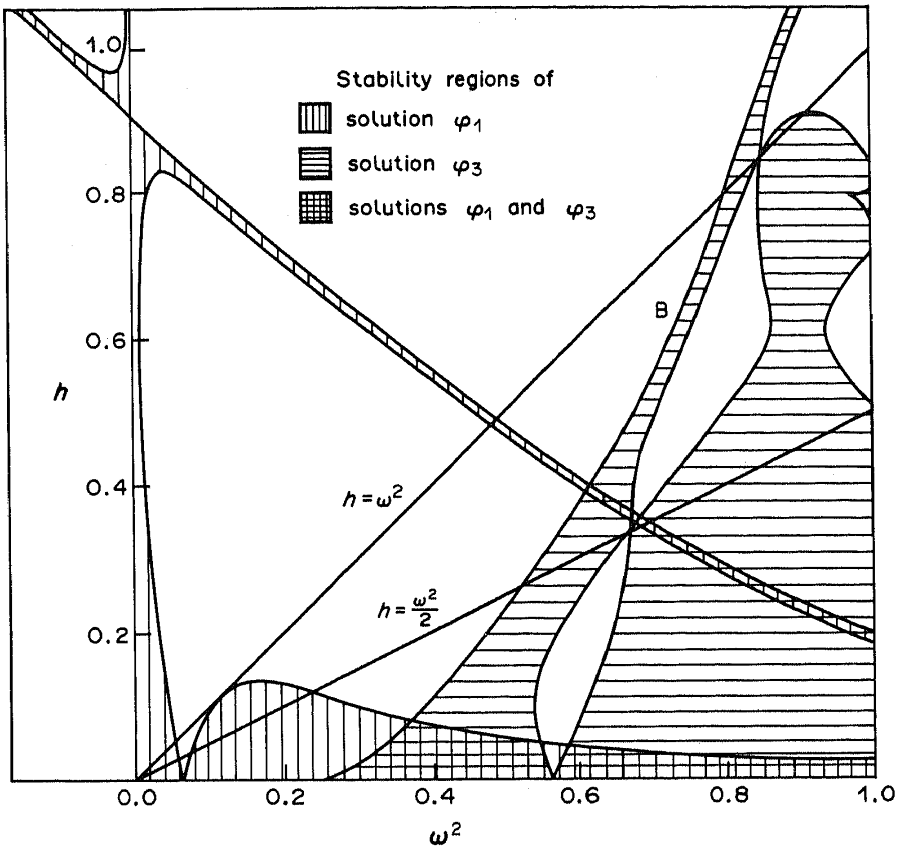

Fig. 9. Stability chart for odd solutions ϕ_1 and ϕ_3.

also instead of the above algorithm for determining the boundaries of parametric resonance zones.

For the operation of numerical integration, the fourth order Runge–Kutta method was used. An absolute error of computations of $\mathrm{tr}\tilde{A}$ was as a rule 10^{-4}.

Let us widen a class of periodic solutions under considerations and study any solutions of the boundary value problem (11) and (12). Before this we shall establish the symmetry properties of eqn (11). It is invariant with the following replacements:

N	t	ϕ	ω^2	h
1	$-t$	$-\phi$	ω^2	h
2	$t+\frac{\pi}{2}$	$\phi-\pi$	$-\omega^2$	h
3	t	$\phi+2\pi$	ω^2	$-h$
4	$t+\pi$	$\phi+2\pi$	ω^2	h
5	$t+2\pi$	ϕ	ω^2	h
6	t	$\phi+4\pi$	ω^2	h

The first three replacements generate the symmetry group of eqn (11). For example, property 4 is formed if we perform replacement 2 twice, and then replacement 3 twice.

Property 6 reflects the fact that the satellite orientation angle δ (see Fig. 1) is determined to within 2π (let us recall that $\phi = 2\delta$).

Property 5 means that the period of eqn (11) is equal to 2π, i.e. to the synodic period of the satellite revolution.

The sense of replacement 3 consists in rotation of the satellite by an angle π. Due to the third property there is no need to consider the case $h < 0$. Obviously, the stability chart for $h < 0$ is the mirror reflection of the stability chart for $h > 0$ with respect to the axis $h = 0$.

Property 2 is equivalent to the rotation of the satellite by angle $\pi/2$; here, in addition to the time shift, the sign of the gravitational parameter changes due to actual redistribution of moments of inertia. Property 2 immediately enables us to find out the existence of periodic solutions which are analogous to ϕ_1, ϕ_2 and ϕ_3 but shifted by π; their stability chart is the mirror reflection of the chart in Fig. 9 with respect to the axis $\omega^2 = 0$. The new shifted solutions will be designated, respectively, by $\phi_{\pi 1}$, $\phi_{\pi 2}$ and $\phi_{\pi 3}$.

All the periodic solutions obtained from the odd solutions ϕ_1, ϕ_2 and ϕ_3 by the above replacements have obviously the property that they are symmetrical with respect to $\phi = \pi k$, $k \in Z$, i.e. the satellite oscillates symmetrically either with respect to its radius vector or with respect to the tangent to orbit. However, a whole set of solutions of the boundary value problem (11) and (12) is not exhausted by the oscillations that have the above property.

5. THE SATELLITE DYNAMICALLY RESEMBLING A SPHERE

Let us consider again eqn (11). We replace the time by $\tau = 2t$ and introduce a new dependent variable $\gamma = (\phi/2) - t - (\pi/2) = \delta + \alpha$. The angle γ is an angle between the line directed to the Sun and the plane of plate S (Fig. 1). We also designate $\epsilon = -\omega^2/2$. Equation (11) written in the new variables has the form

$$\frac{d^2\gamma}{d\tau^2} + h|\sin\gamma|\cos\gamma = \epsilon\sin(\tau - 2\gamma). \quad (14)$$

The boundary conditions (12) for eqn (11) are transformed into the ones

$$\gamma(0) = \gamma(4\pi) - 2\pi, \quad \frac{d\gamma}{d\tau}(0) = \frac{d\gamma}{d\tau}(4\pi) \quad (15)$$

for eqn (14).

The function $Q(\gamma) \equiv h|\sin\gamma|\cos\gamma$ in eqn (14) is 2π-periodic by its argument and has the zero average value on period.

Let us determine the potential

$$V \equiv \int Q(\gamma)\,d\gamma = h/2|\sin\gamma|\sin\gamma$$

and the energy $E = \frac{1}{2}(d\gamma/d\tau)^2 + V$.

We shall consider eqn (14) with small values of ϵ: $|\epsilon| \ll 1$. It corresponds to the satellite which dynamically resembles a sphere. If $\epsilon = 0$ the problem has the integral of energy $E = \text{const}$.

We introduce the rotation phase

$$\psi = (2\pi/T_0)\tau + \psi_0, \quad \psi_0 = \text{const}.$$

The $T_0(E)$ is the period which can be calculated by the formula

$$T_0(E) = \int_0^{2\pi} \frac{d\gamma}{\sqrt{2[E - V(\gamma)]}}.$$

The boundary value problem (14) and (15) corresponds to the resonant rotational regime of motion with the period

$$T_0(E) = 4\pi. \quad (16)$$

When studying this regime we shall follow[10]. The meaning of eqn (16) is that its root is such an energy level at which 4π-periodical rotation (15) exists.

In (14) we shall pass from the variables τ, γ, $d\gamma/d\tau$ to the new variables τ, E, ψ. Equation (14) written in these new variables is equivalent to the system

$$\frac{dE}{d\tau} = \epsilon\mathscr{E}(E, \gamma, \tau),$$

$$\frac{d\psi}{d\tau} = \frac{2\pi}{T_0(E)} + \epsilon\Psi(E, \gamma, \tau), \quad (17)$$

where

$$\mathscr{E} = \sqrt{2[E - V(\gamma)]}\sin(\tau - 2\gamma)$$

and

$$\Psi = -\frac{2\pi}{T_0}F\left[\frac{1}{2[E - V(\gamma)]}\right]\mathscr{E}.$$

The linear operator $F[\chi]$, $\chi = \chi(E, \gamma)$ is determined as follows:

$$F[\chi] \equiv \int_0^\gamma \frac{\chi\,d\eta}{\sqrt{2[E - V(\eta)]}} - \frac{1}{T_0}\left(\int_0^{2\pi} \frac{\chi\,d\eta}{\sqrt{2[E - V(\eta)]}}\right)\int_0^\gamma \frac{d\eta}{\sqrt{2[E - V(\eta)]}}.$$

Introducing the variable

$$y = \tau - 2\psi = \tau(1 - 4\pi/T_0) - 2\psi_0$$

allows one to pass from (17) to the new system:

$$\frac{dE}{d\tau} = \epsilon\mathscr{E}(E, \gamma, y + 2\psi),$$

$$\frac{dy}{d\tau} = \lambda(E) - 2\epsilon\Psi(E, \gamma, y + 2\psi),$$

$$\frac{d\psi}{d\tau} = \frac{2\pi}{T_0(E)} + \epsilon\Psi(E, \gamma, y + 2\psi), \quad (18)$$

where

$$\lambda(E) = 1 - 4\pi/T_0(E).$$

Obviously (16) is equivalent to the equation

$$\lambda(E) = 0. \quad (19)$$

We average eqns (18) by ψ along the trajectory of the degenerated system ($\epsilon = 0$), $\gamma = \gamma(E, \psi)$ with $E = E_0$. For convenience of computations we choose a representation of the solution $\gamma = \gamma(E, \psi)$ such that $\gamma(E, \psi_0) = 0$.

Let us average the right-hand part of the first equation of (18):

$$\bar{\mathscr{E}}(E_0, y) = \frac{1}{2\pi}\int_{\psi_0}^{\psi_0 + 2\pi} \mathscr{E}(E_0, \gamma(E_0, \psi), y + 2\psi)\,d\psi$$

$$= \frac{1}{T_0(E_0)} \times \int_{\tau_0}^{\tau_0 + T_0(E_0)} \mathscr{E}\left(E_0, \gamma\left(E_0, \frac{2\pi}{T_0(E_0)}\tau + \psi_0\right), y + \frac{4\pi}{T_0(E_0)}\tau + 2\psi_0\right)d\tau. \quad (20)$$

Along the trajectory of the degenerated system we have $d\tau = \{2[E_0 - V(\gamma)]\}^{-1/2}\,d\gamma$. By taking into account the last relation we obtain from (20)

$$\bar{\mathscr{E}}(E_0, y) = \frac{1}{T_0(E_0)}\int_0^{2\pi} \mathscr{E}\left(E_0, \gamma, y + \frac{4\pi}{T_0(E_0)} \times \int_0^\gamma \frac{d\eta}{\sqrt{2[E_0 - V(\eta)]}}\right)\frac{d\gamma}{\sqrt{2[E_0 - V(\eta)]}}.$$

The equation for $y = y_0$ corresponding to the stationary resonant regime is

$$\bar{\mathscr{E}}(E_0, y) = 0. \quad (21)$$

Thus, by solving first (19) and then (21) we can determine values of both E_0 and y_0.

Now we shall engage ourselves in determining specific forms of functions λ and $\mathscr{E}$. By introducing the function

$$S(E,\gamma) \equiv \int_0^{\gamma} \frac{d\gamma}{\sqrt{2[E - V(\gamma)]}} \tag{22}$$

and taking into account (16) which can be rewritten in the form

$$S(E, 2\pi) = 4\pi, \tag{23}$$

we transform (21) to the equation

$$\int_0^{2\pi} \sin(y + S(E_0, \gamma) - 2\gamma)\, d\gamma = 0. \tag{24}$$

The rotation phase corresponding to the stationary regime is $\psi = \tau/2 - y^*/2$. Then we obtain that for $h = h^*$ there is a continuum of solutions of the boundary value problem (14) and (15) that differ from each other by a shift of time τ.

In order to answer the question if there is a critical value h^* we should first write down (22) for $S(E,\gamma)$ in an expanded form by substituting an explicit expression for $V(\gamma)$ into (22). By virtue of validity of the relation $S(E, \gamma + 2\pi n) = nS(E, 2\pi) + S(E, \gamma)$, $n \in Z$, it is sufficient to find $S(E, \gamma)$ only for $\gamma \in [0, 2\pi]$. Omitting intermediate calculations we give the final formula for the function $S(E,\gamma)$:

$$S(E,\gamma) = \begin{cases} \dfrac{k_+}{\sqrt{h}} F(\gamma, k_+), & 0 \leqslant \gamma \leqslant \dfrac{\pi}{2}, \\ \dfrac{k_+}{\sqrt{h}} [2K(k_+) - F(\pi - \gamma, k_+)], & \dfrac{\pi}{2} < \gamma \leqslant \pi, \\ \dfrac{2k_+}{\sqrt{h}} K(k_+) + \dfrac{k_-}{\sqrt{h}} \left[K(k_-) - F\left(\dfrac{3\pi}{2} - \gamma, k_-\right)\right], & \pi < \gamma \leqslant \dfrac{3\pi}{2}, \\ \dfrac{2k_+}{\sqrt{h}} K(k_+) + \dfrac{k_-}{\sqrt{h}} \left[K(k_-) + F\left(\gamma - \dfrac{3\pi}{2}, k_-\right)\right], & \dfrac{3\pi}{2} < \gamma \leqslant 2\pi. \end{cases} \tag{27}$$

We designate

$$a = a(E_0) \equiv \int_0^{2\pi} \cos(S(E_0, \gamma) - 2\gamma)\, d\gamma,$$

$$b = b(E_0) \equiv \int_0^{2\pi} \sin(S(E_0, \gamma) - 2\gamma)\, d\gamma.$$

Then eqn (24) can be written as

$$a \sin y + b \cos y = 0. \tag{25}$$

If $a^2 + b^2 \neq 0$, there is a unique series of solutions (25)

$$y_0 = y_0^* + \pi k, \quad k \in Z,$$

$$y_0^* = \begin{cases} -\operatorname{arctg}\dfrac{b}{a}, & a \neq 0, \\ \dfrac{\pi}{2}, & a = 0. \end{cases} \tag{26}$$

From a definition of the variable y with involvement of (16) the relation $y_0 = -2\psi_0$ follows. Hence, it is easy to obtain that in a general case, i.e. when $a^2 + b^2 \neq 0$, the rotation phase is determined to within $\pi/2$: $\psi = (\tau/2) - (y_0^*/2) + (\pi/2)n$, $n \in Z$. From the last equality it is seen that a series of stationary phases (26) corresponds to the shift of time τ by πn, $n \in Z$. The shift of τ by π is equivalent to replacement 2 from point 3. The shift by 2π (replacement 4) changes, in fact, nothing. Therefore, we can conclude that if $a^2 + b^2 \neq 0$ there are two different stationary rotations (they correspond to symmetrical solutions ϕ_1 and $\phi_{\pi 1}$).

We have not yet considered the possibility of simultaneous vanishing of a and b for a certain h (it means also E_0). So let there exist $h = h^*$ such that $a = b = 0$. Then any $y = y^*$ satisfies eqn (25).

Here $k_+^2 \equiv h/2E$, $k_-^2 \equiv h/(2E + h)$; the moduli k_+ and k_- are connected by the relation:

$$1/k_-^2 - 1/k_+^2 = 1; \quad F(\zeta, k) = \int_0^{\zeta} \frac{d\vartheta}{\sqrt{1 - k^2 \sin^2 \vartheta}}$$

is an incomplete elliptical integral of the first kind;

$$K(k) = F\left(\frac{\pi}{2}, k\right)$$

is a complete elliptical integral of the first kind.

The rotational motions of interest exist when $E > h/2$ (with $\epsilon = 0$). Then $k_+^2 < 1$, $k_-^2 < 1/2$. Hence, expressions (27) are correct.

Note that eqn (23) can be written in the form:

$$2\pi\sqrt{h} = k_+ K(k_+) + k_- K(k_-).$$

Knowing the function $S(E,\gamma)$ we can calculate numerically the functions $a(h)$ and $b(h)$ (Fig. 10). It is seen that there is a unique resonant value h^* (at least for not very large h) for which $a = b = 0$. The critical value h^* can be calculated with high accuracy: $h^* = 0.90833$. It is the value of h for which the stability zone boundary of solution ϕ_1 (and $\phi_{\pi 1}$) crosses the axis h. Indeed, let us consider the problem of stability of the obtained stationary regimes. According to[10] the stationary resonant regime (19) and (21) is stable if the conditions

$$\frac{d\lambda}{dE} \cdot \frac{\partial \mathscr{E}}{\partial y_0}(E_0, y_0) < 0, \tag{28}$$

$$\frac{\partial \mathscr{E}}{\partial E}(E_0, y_0) - 2\frac{\partial \Psi}{\partial y_0}(E_0, y_0) < 0 \tag{29}$$

are valid.

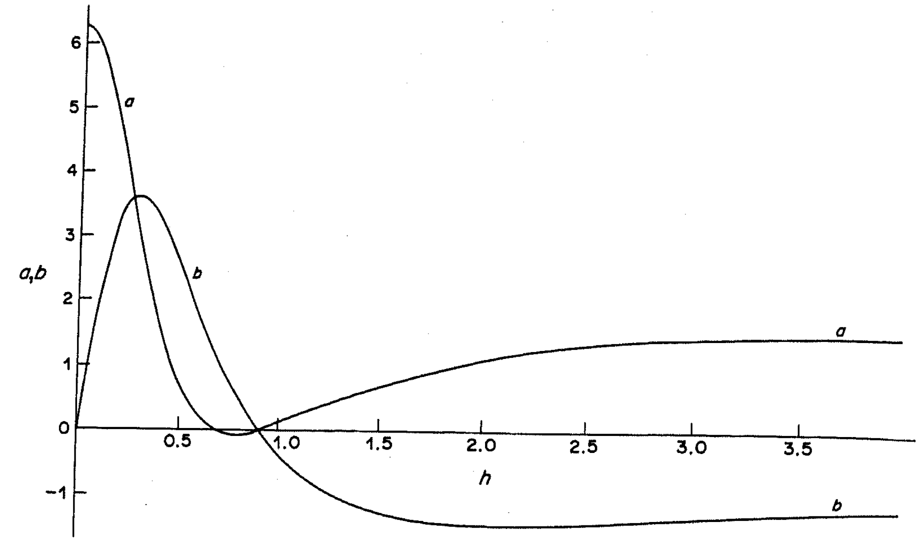

Fig. 10. Integrals a and b as functions of light parameter h.

The averaging of Ψ is carried out similarly to the function $\mathscr{E}$.

With appropriate calculations we can show that the expression in the left-hand side of (29) vanishes and, hence, condition (29) is not satisfied, i.e. the approximation constructed is not sufficient for determining stability. However, if we add an arbitrary small viscous friction in the original eqn (14) then condition (29) will prove to be satisfied. As for condition (28), we can show, first, that always $d\lambda/dE_0 < 0$ and, second, that $\delta\bar{\mathscr{E}}/\partial y_0 = (a \cos y_0 - b \sin y_0)/T_0(E_0)$. By comparing (28), the last inequality and relation as well as the expressions for a series of solutions (26) we conclude that the passage through the critical value h^* leads to a change of stability of solutions ϕ_1 and $\phi_{\pi 1}$, which agrees with the chart in Fig. 9.

However, from this figure it is seen that the stability region boundary crosses the axis h at the critical point h^* not along the normal. By taking into account that for $h = h^*$, $\omega^2 = 0$ there is a continuum of solutions and we should expect that this set can be found outside the axis $\omega^2 = 0$. It was found and studied by using numerical techniques similar to those described in point 3. These algorithms are different, however, in that the two-parametric boundary value problem of searching for arbitrary periodic oscillations instead of a one-parametric boundary value problem of searching for odd solutions was solved.

The stability chart of the solutions under considerations, constructed by such a method, is given in Fig. 11. The boundaries of the stability and instability regions are determined not only for small values of ω^2 but over the whole interval of variation of gravitational parameter as in Fig. 9 (due to the symmetry property of solutions it is sufficient to consider the region $\omega^2 > 0$). Let us begin describing Fig. 11 from the bifurcation curve $\tilde{B}$—a boundary of the stability region of solution ϕ_1 (cf. Fig. 9) that crosses the point $\omega^2 = 0$, $h = h^*$. The bifurcation curve $\tilde{B}_\pi$—a mirror reflection of the curve $\tilde{B}$ (Fig. 9) with respect to the axis h—is a boundary of the stability region of solution $\phi_{\pi 1}$. In the triangular region bounded by these curves, in addition to symmetrical solutions ϕ_1 and $\phi_{\pi 1}$ there are nonsymmetrical ones which have been found at the point $\omega^2 = 0$, $h = h^*$. These solutions have the stability region adjoining the bifurcation curves $\tilde{B}$ and $\tilde{B}_\pi$. The nonsymmetrical solutions are similar to nonsymmetrical eccentricity oscillations of the satellite described in[11].

Let us introduce a new characteristic of the periodic solution—the average value over the period:

$$\bar{\phi} \equiv \frac{1}{2\pi} \int_0^{2\pi} \phi(\alpha)\, d\alpha.$$

Now we shall see how this value varies with h if ω^2 is constant. Let $\omega^2 = 0.098$. Figure 12 shows the function $\bar{\phi}(h)$ obtained numerically for nonsymmetrical solutions. The curve is symmetrical with respect to the axis h. At the point B_1 that belongs to the boundary $\tilde{B}$ (see Fig. 11) a pair of nonsymmetrical solutions is generated from the solution ϕ_1, their average value moduli increase with h. At the point B_2 corresponding to the curve $\tilde{B}_\pi$ this pair disappears. The straight line $\bar{\phi}_{\pi 1} = \pi$ corresponds to the solution $\phi_{\pi 1}$.

The existence and stability regions of all solutions obtained are shown in the vicinity of the resonant point $\omega^2 = 0$, $h = h^*$ in the form of a three-dimensional diagram in Fig. 13. The diagram somehow combines Figs 11 and 12.

6. EXAMPLE: A PLANE UMBRELLA

As a specific example we consider a satellite model that in size and mass characteristics is close to a usual umbrella against rain or sun.

Let the satellite be modelled by a plane disk with the radius R, the mass m_d and the orthogonal

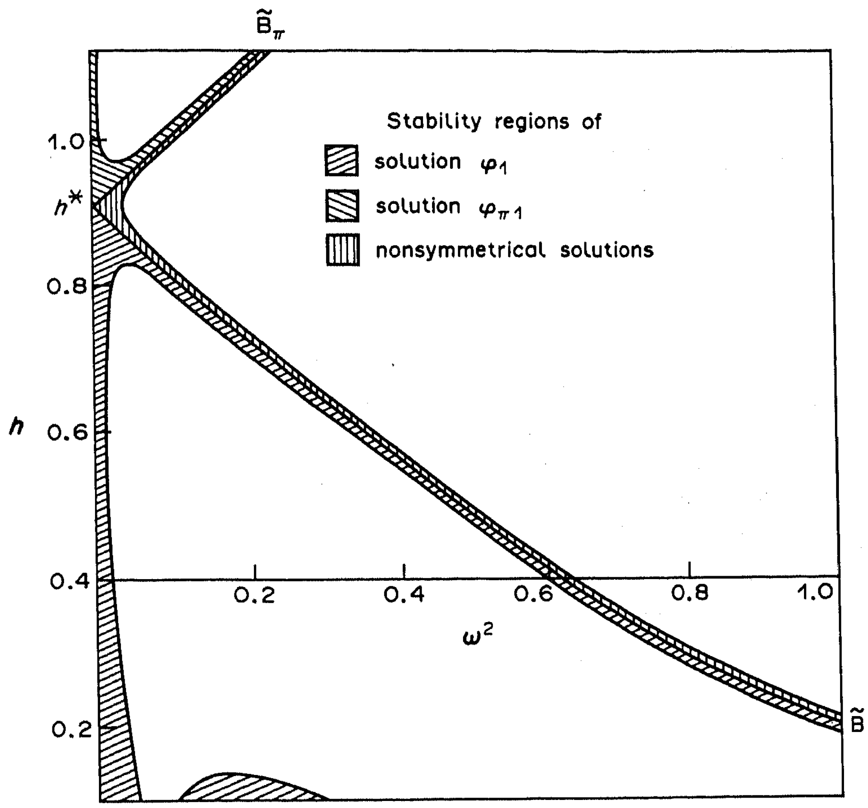

Fig. 11. Stability chart for symmetrical and nonsymmetrical solutions.

homogeneous rod of mass m_s that is fixed at the centre of the disk (Fig. 14). We assume that the centre of mass of the system is located at the distance r from the disk centre, while the length of the rod is $4r$. Let the moment of inertia of the disk (and the satellite) with respect to the rod axis be $A = m_d r_*^2$, where r_* is the inertia radius of the disk.

The satellite's moment of inertia with respect to the axis crossing its centre of mass and orthogonal to the rod is calculated by the formula

$$B = \frac{1}{2} m_d r_*^2 + m_d r^2 + \frac{1}{3}\left(\frac{m_s}{4} r^2 + \frac{3m_s}{4}(3r)^2\right)$$
$$= \tfrac{1}{2} m_d r_*^2 + \tfrac{1}{3}(7M - 4m_d)r^2.$$

Let us give the following numerical values: $R = 0.45$ m, $r = 0.16$ m, $r_* = 0.29$ m (the inertia radius of the homogeneous disk $r_{**} = R/\sqrt{2}$,

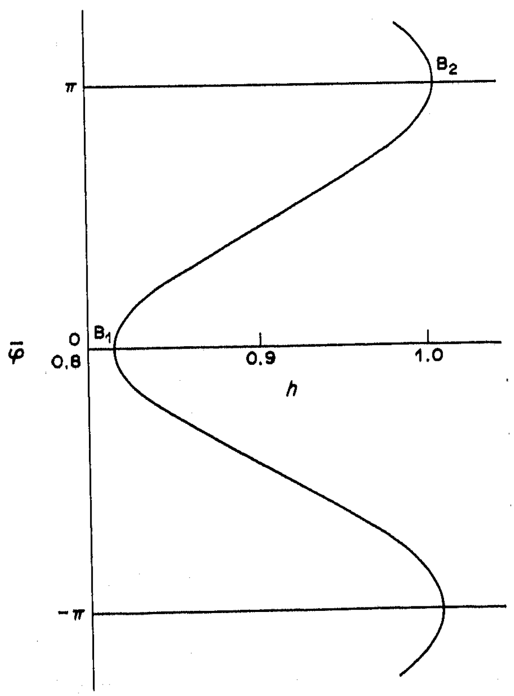

Fig. 12. The average angle $\bar{\varphi}$ as a function of light parameter h for nonsymmetrical solutions at $\omega^2 = 0.098$.

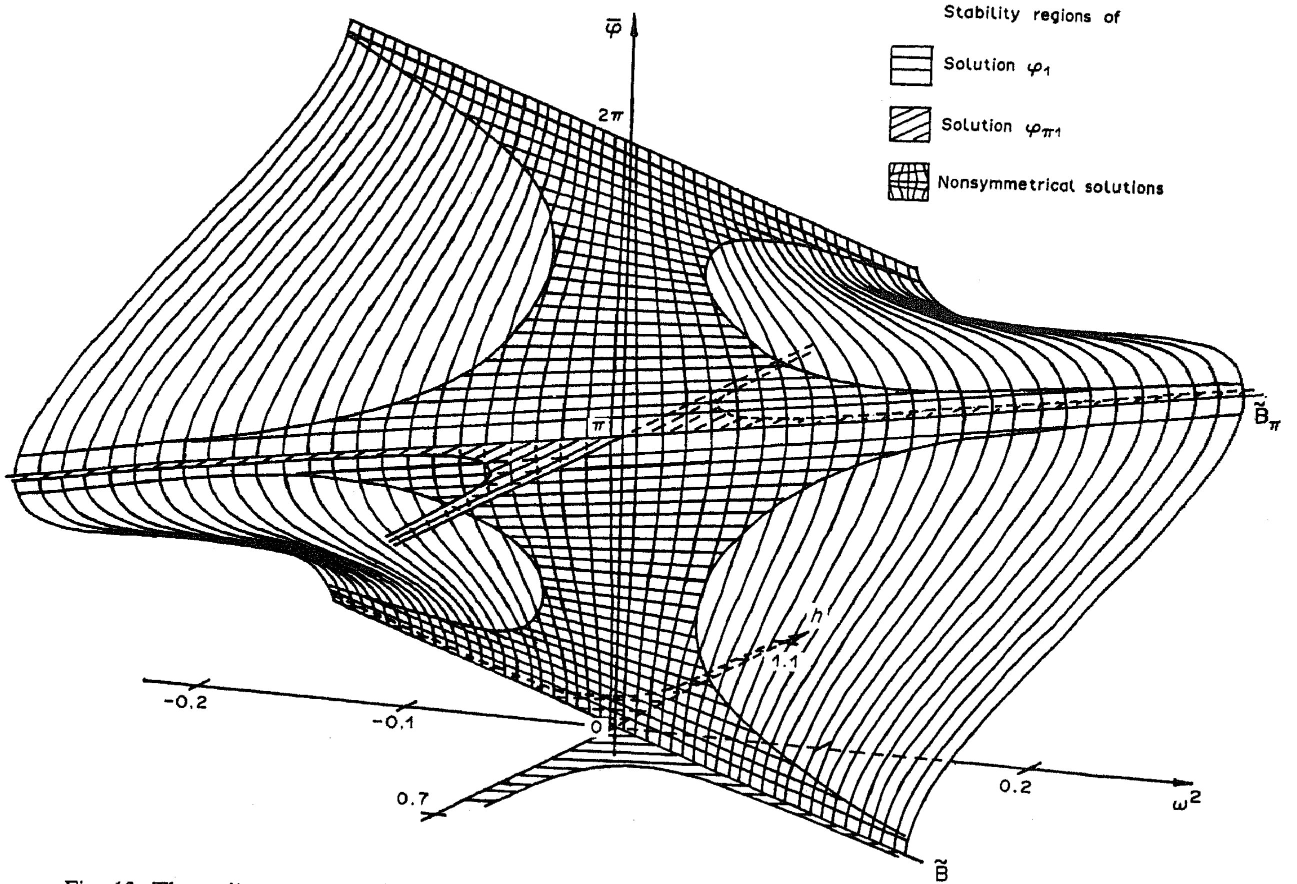

Fig. 13. Three-dimensional diagram of existence and stability regions for all solutions in the vicinity of the resonant point $\omega^2 = 0$, $h = h^*$.

$r_{**} = 0.32$ m, i.e. $r_* = 0.29$ m corresponds to a slightly inhomogeneous disk).

A total mass of the satellite $M = m_d + m_s$ is taken equal to $M = 0.476$ kg while the disk mass $m_d = 0.4$ kg. Then $A = 3.364 \times 10^{-2}$ kg m^2, $B = 3.160 \times 10^{-2}$ kg m^2. Gravitational parameter is $\omega^2 = \frac{3}{4}(1 - A/B) = -0.048$. The light parameter is calculated by the formula $h = \frac{1}{4}(p_c/\mu)R_0^3(Sr/B)(1 - \epsilon_c)$, where μ is the gravitational constant (for the Earth $\mu = 3.986 \times 10^{14}$ m^3/c^2), R_0 is the radius of the satellite orbit, $S = \pi R^2$ is the area of one side of the disk.

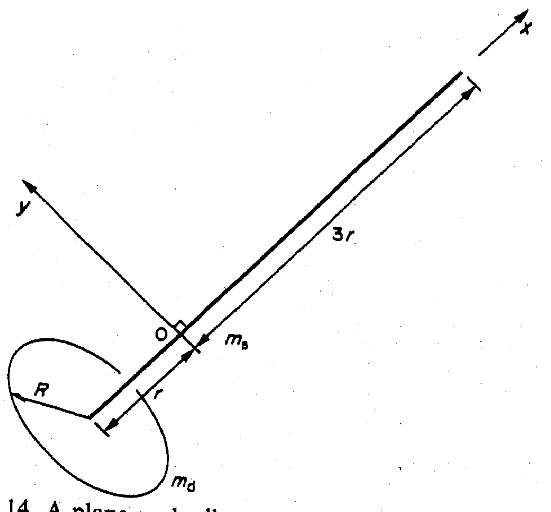

Fig. 14. A plane umbrella as an example of the satellite.

We assume that the satellite orbit is geocentric circular of the $R_0 = 15{,}078$ km, and the reflection coefficient of the disk surface $\epsilon_c = 0.97$, i.e. the disk is almost mirror-like. Under these assumptions the light parameter $h = 0.964$.

The periodic solution for such parameter values is shown in Fig. 7. As it is seen from the chart in Fig. 9 this solution is stable despite the fact that the satellite oscillates in such a way that in its average position the axis with a maximal moment of inertia is directed along the radius vector. In the absence of light perturbations the oscillations of that kind are unstable[3].

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