

Weighing matrices and optical quantum computing

Steven T Flammia¹ and Simone Severini²

¹ Perimeter Institute for Theoretical Physics, 31 Caroline St. N, Waterloo, Ontario, N2L 2Y5, Canada

² Institute for Quantum Computing and Department of Combinatorics & Optimization, University of Waterloo, 200 University Ave. W, Waterloo, Ontario, N2L 3G1, Canada

E-mail: sflammia@perimeterinstitute.ca and simoseve@gmail.com

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Abstract

Quantum computation in the one-way model requires the preparation of certain resource states known as cluster states. We describe how the construction of continuous-variable cluster states for optical quantum computing relate to the existence of certain families of matrices. The relevant matrices are known as weighing matrices, with a few additional constraints. We prove some results regarding the structure of these matrices, and their associated graphs.

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1. Introduction

In the standard model of quantum computing [1], the computation proceeds by coherently performing unitary dynamics on a simple initial state of a quantum system before being read out by a sequence of local measurements. By contrast, the one-way model of quantum computing [2] eliminates the need for coherent quantum evolution, substituting instead a highly entangled, but easily prepared, initial state as the resource for the computation. The typical resource state used is the *cluster state* [3], whose preparation consists of, for example, coupling a system of qubits with a nearest-neighbor Ising interaction. The computation proceeds by a sequence of local measurements on the cluster state and the basis chosen at each step determines the computation. In this way, a cluster state is like a quantum breadboard on which the quantum circuit is inscribed via measurements.

Although cluster states were originally defined using qubits, they generalize to d -level quantum systems [4] and continuous variables [5, 6]. Recently, Menicucci *et al* [7] proposed a method for efficiently generating continuous-variable cluster states (CVCS). The method relies on pumping an optical cavity containing a special nonlinear medium at certain carefully chosen frequencies. The resulting CVCS is encoded in the quadratures of the photons that emerge from the cavity.

The method of CVCS generation proposed in [7] required, on grounds of experimental and theoretical tractability, the existence of certain families of matrices. The required matrices are *weighing matrices*, defined as matrices W whose entries are in $\{0, \pm 1\}$ and satisfy $W W^T = kI$, where k is called the *weight*. The weighing matrices were obliged to have a *Hankel* form, meaning that the skew-diagonals of the matrix were all constant, and any skew-diagonal with support on the main diagonal must be identically 0. A matrix with vanishing main diagonal is said to be *hollow*. As we show later, when viewing the matrices as adjacency matrices for an associated graph, this is implied when the graph is connected and bipartite.

In this paper, we investigate the mathematical structure behind the construction in [7] and begin to classify the associated matrices. The discussion is organized as follows. In section 2, we give a brief glimpse of the physics which motivates the restrictions given in the previous paragraph. We begin the mathematical discussion in section 3, where we discuss anticirculant graphs, and section 4, where we discuss anticirculant weighing matrices, a special case of the Hankel weighing matrices. In section 5, we complete the rather trivial classification of the case of weight 2, while section 6 provides several examples about the case of weight 4. We discuss a number of open problems in section 7, and conclude in section 8.

2. Quantum computing in the frequency comb

In this section, we outline the physics that naturally leads us to consider Hankel hollow weighing matrices. For a more detailed discussion of the physics, see [7–10].

The resonant frequencies of an optical cavity are defined classically as the modes which constructively interfere inside the cavity. Because they are evenly spaced, they form a so-called a frequency comb. By placing a nonlinear medium inside the cavity, pump photons with frequency ω_p can downconvert into an entangled pair of photons, so long as the new photons' frequencies satisfy an energy conservation constraint,

$$\omega_p = \omega_m + \omega_n, \quad (1)$$

where ω_m and ω_n are the m th and n th resonant frequencies of the cavity. The range over which the downconversion can occur is limited by the phasematching bandwidth of the cavity.

More generally, multiple frequencies can be simultaneously phasematched inside the cavity, and upconversion is allowed as well as downconversion. The total Hamiltonian is then idealized by the following:

$$\mathcal{H} = i\hbar\kappa \sum_{p \in P} \sum_{m+n=p} M_{mn} (\hat{a}_m^\dagger \hat{a}_n^\dagger - \hat{a}_m \hat{a}_n), \quad (2)$$

where P is the (discrete) spectrum of the polychromatic pump, κ is a global coupling strength, \hat{a}_n^\dagger is the creation operator for the n th mode, and M_{mn} are matrix elements of a symmetric matrix. In units of the fundamental frequency of the cavity, the restriction of the inner sum to terms with $m + n = p$ enforces photon energy conservation, and the magnitude of M_{mn} determines the strength of the coupling in units of $\hbar\kappa$, while the sign determines whether the photons are upconverted or downconverted. If all of the coupling strengths are equal, then M is a Hankel matrix whose elements are either 0 or $\pm c$, where c is a constant. Experimental simplicity further demands that there is no single-mode squeezing and hence we require that the elements along the main diagonal of M are all zero, i.e. M must be a hollow matrix.

The Hamiltonian (2) can be used to create continuous-variable cluster states, defined as Gaussian states satisfying the relation

$$\mathbf{p} - A\mathbf{q} \rightarrow \mathbf{0}, \quad (3)$$

where \mathbf{p} and \mathbf{q} are vectors of amplitude and phase quadratures for the k th cavity mode, $q_k = \hat{a}_k^\dagger + \hat{a}_k$ and $p_k = i(\hat{a}_k^\dagger - \hat{a}_k)$, A is the symmetric weighted adjacency matrix of the graph of the CVCS, and the arrow denotes the limit of large squeezing. Reference [8] first proved a nonunique relationship between the adjacency matrix A , which describes the cluster state, and the adjacency matrix M , which describes the coupling between the modes inside the cavity. The general relationship is somewhat complicated, so references [7, 9, 10] introduced the simplifying ansatz that

$$AA^T = A^2 = 1, \tag{4}$$

from which it follows [9, 10] that (up to a trivial relabeling)

$$M = A. \tag{5}$$

In short, assuming A is an orthogonal matrix implies the graph describing the couplings between photons in the cavity (M) is *identical* to the graph describing the cluster state (A), which is a vast simplification over the general relationship. The derivation requires that M be the adjacency matrix of a bipartite graph, but as we will see, one can derive this from the Hankel and hollow constraints.

We now see how the twin demands of experimental and theoretical simplicity for creating CVCS in an optical cavity naturally lead us to consider Hankel hollow orthogonal matrices, all of whose nonzero elements are either $\pm c$, for some constant c . We can of course consider renormalized matrices instead by dividing out c , so that all entries are ± 1 ; after finding these renormalized matrices, we can then reintroduce the constant afterward so that (4) holds. Thus, we see that finding and classifying Hankel hollow weighing matrices is our primary interest.

For completeness, we mention one further elaboration that is possible for this scheme. If the photons' polarization, transverse, or spatial degrees of freedom are taken into account, then the Hamiltonian (2) can be modified by adding additional indices for these modes. If these additional modes are all frequency degenerate, then an additional sum over these degenerate modes appears in (2), which allows for the symmetric intercoupling between the degenerate degrees of freedom. If such an interaction could be simultaneously phasematched by the nonlinear medium inside the cavity, then one is naturally lead to consider *block*-Hankel matrices, where the size of the blocks is equal to the number of degenerate degrees of freedom. We leave the consideration of block-Hankel hollow weighing matrices to future work.

3. Anticirculant graphs

In this section, we introduce the definition of anticirculant graphs and study some of their first properties. For the sake of self-containedness, we will define here all the graph-theoretic concepts we require. For further background on theory of graphs the interested reader is referred to the book by Diestel [11]. Our reference about permutations and finite groups is Cameron [12].

We will work with simple graphs. A (simple) *graph* $G = (V, E)$ is an ordered pair of sets defined as follows: $V(G)$ is a non-empty set, whose elements are called *vertices*; $E(G)$ is a non-empty set of unordered pairs of vertices, whose elements are called *edges*. An edge of the form $\{i, i\}$ is called a *loop*. Two vertices i and j are said to be *adjacent* if $\{i, j\} \in E(G)$. Often, this is simply denoted by writing ij . A graph is *loopless* if it has no loops. The *adjacency matrix* of a graph G is denoted by $A(G)$ and defined by

$$[A(G)]_{i,j} := \begin{cases} 1, & \text{if } ij \in E(G); \\ 0, & \text{if } ij \notin E(G). \end{cases}$$

An adjacency matrix of a loopless graph is said to be *hollow*. The *degree* of a vertex i in a graph G is the number of edges incident with the vertex, that is, $\#\{j : ij \in E(G)\}$. A graph is *regular* if each of its vertices has the same degree. We say d -*regular* to specify that each vertex has degree d . A square matrix is *Hankel* if it has constant skew-diagonals. These are diagonals that traverse the matrix from North-East to South-West. (For this reason, these are also called *antidiagonals*). Note that an $n \times n$ matrix has $2n - 1$ skew-diagonals. An $n \times n$ Hankel matrix is said to be *anticirculant* (or, equivalently, *skew-circulant* or *backcirculant*) if the i th and $(i + n)$ th skew diagonals are equal. A *permutation* of length n is a bijection $\pi : [n] \rightarrow [n]$, where $[n] = \{1, 2, \dots, n\}$. A *permutation matrix* of dimension n is an $n \times n$ matrix P with the following two properties: $[P]_{i,j} \in \{0, 1\}$ for every i, j ; P has a unique 1 in each row and in each column. A permutation π is said to *induce* a permutation matrix P if $[P]_{i,\pi(i)} = 1$ for every i .

Two graphs G and H are *isomorphic* if there is a permutation matrix P such that $A(G) = PA(H)P^{-1}$. We write $G \cong H$ to denote that graphs G and H are isomorphic. A graph G is *bipartite* if there is a permutation matrix P such that

$$PA(G)P^{-1} = \begin{pmatrix} 0 & M \\ M^T & 0 \end{pmatrix}, \tag{6}$$

for some matrix M . If $M = J_n$, where J_n is the all-ones matrix of size n , then $G \cong K_{n,n}$, the *complete bipartite graph*. A graph is said to be *anticirculant* if it is isomorphic to a graph whose adjacency matrix is anticirculant. Let us denote by $\mathcal{L}_{n,d}$ and by \mathcal{L}_n the set of all d -regular anticirculant graphs on n vertices and the set of anticirculant graphs on n vertices, respectively. Note that $\#\mathcal{L}_{n,d} = \binom{n}{d}$. Therefore,

$$\#\mathcal{L}_n = \sum_{d=1}^{n-1} \#\mathcal{L}_{n,d} = \sum_{d=1}^{n-1} \binom{n}{d} = 2^n - 2.$$

Given $G \in \mathcal{L}_n$, the ordered set $S(G) = (s : [A(G)]_{1,s} = 1)$ is said to be the *symbol* of G , in analogy with the terminology used for circulant graphs. Indeed, $S(G)$ specifies G completely. In cycle notation, each element $s \in S(G)$ is associated with a permutation of the form

$$\pi_s = (1, s)(2, s - 1) \cdots (s + 1, n)(s + 2, n - 1) \cdots, \tag{7}$$

where the subtraction is modulo n . In line notation, $\pi_s = (s)(s - 1)(s - 2) \cdots (n)(n - 1) \cdots (s + 1)$. On the basis of the definitions, we can collect the facts below:

Proposition 1. *Let $G \in \mathcal{L}_n$ be a graph. The following statements hold true:*

- (i) *The graph G is d -regular with $d = \#S(G)$.*
- (ii) *If G is loopless then n is even.*
- (iii) *If G is loopless then each $s \in S(G)$ is even.*
- (iv) *The graph G is bipartite.*

Proof. In order:

- (i) Let π_s be the permutation associated with the element $s \in S(G)$. Then π_s induces a symmetric permutation matrix P_s . Since each P_s is defined by s , we have $[P_s]_{i,j} = 1$ if and only if $[P_t]_{i,j} = 0$ for all $t \neq s$. Consequently, $A(G) = \sum_{s \in S(G)} P_s$, and G is a regular graph with degree $d = \#S$.
- (ii) If the size of P_s is odd, then $[P_s]_{i,i} = 1$ for some i . This is because π_s is necessarily an involution of odd length, and for this reason it has at least one fixed point. (Recall that an *involution* is a permutation that is its own inverse.)

- (iii) Suppose on the contrary that $s = 2k + 1$, for some $k \geq 1$. Then, $[A(G)]_{1,s} = [A(G)]_{1,2k+1} = [A(G)]_{k+1,k+1} = 1$, contradicting the hypothesis of G being loopless.
- (iv) By the form of each π_s (see equation (7)), it is sufficient to observe that vertices labeled by odd numbers are adjacent only to vertices labeled by even numbers and *viz.* \square

At this stage, we are ready to prove that the members of \mathcal{L}_n are Cayley graphs. The Cayley graph $G = G(\Gamma, T)$ of a group Γ w.r.t. the set $T \subseteq \Gamma$ is the graph in which $V(G) = \{\Gamma\}$ and $\{g, h\} \in E(G)$, if there is $s \in T$ such that $gs = h$. The dihedral group D_{2k} of order $2k$ is the group of symmetries of a regular k -gon. The dihedral group D_{2k} is non-Abelian and presented as

$$D_{2k} = \langle s, t : t^k = e, s^2 = e, sts = t^{-1} \rangle,$$

where t is a rotation and s is a reflection. The elements of D_{2k} are k rotations $t^0, t, t^2, \dots, t^{k-1}$ and k reflections $s, st, st^2, \dots, st^{k-1}$. A graph G is said to be *connected* if there is no permutation matrix P such that $PA(G)P^{-1} = \bigoplus_i M_i$, for some matrices M_i . If a Cayley graph of a group Γ is not connected then it is the disjoint union of isomorphic Cayley graphs of a subgroup of Γ , with each isomorphic component corresponding to a coset of the subgroup.

Proposition 2. *Let G be a loopless anticirculant graph. Then G is a Cayley graph of the dihedral group with respect to a set of reflections.*

Proof. The adjacency matrix of a Cayley graph $G = G(\Gamma, T)$ can be written as $A(G) = \sum_{t \in T} \rho_{\text{reg}}(t)$, where ρ_{reg} is the regular permutation representation of Γ . When the order of Γ is n , this is an homomorphism of the form $\rho_{\text{reg}} : \Gamma \rightarrow \Sigma_n$, where Σ_n is the set of all $n \times n$ permutation matrices. Each $\rho_{\text{reg}}(t)$ describes the action of the group element t on the set $\{1, 2, \dots, n\}$. Let $G \in \mathcal{L}_n$ be a loopless graph. By proposition 1, we need to take $n = 2k$. Let us label the lines (rows and columns) of $A(G)$ with the group elements of D_n , of order $n = 2k$, in the following order:

$$t^0 = e \quad s \quad t \quad st \quad t^2 \quad st^2 \quad \dots \quad t^{k-1} \quad st^{k-1},$$

where e is the identity element of the group. Consider $\rho_{\text{reg}}(st^l)$, where $1 \leq l \leq k$. By applying the generating relations of D_n , it is straightforward to verify that $\rho_{\text{reg}}(s) = P_2$, and $\rho_{\text{reg}}(st^l) = P_{2+2l}$, when $1 \leq l \leq k - 1$. The elements of $S(G)$ are then associated with reflections in D_{2k} ,

$$2 \rightarrow s \quad 4 \rightarrow st \quad 6 \rightarrow st^2 \quad \dots \quad 2k \rightarrow st^{k-1}.$$

This concludes the proof of the proposition. \square

An example is useful to clarify this result. Let \mathbb{Z}_n be the additive group of integers modulo n . The Cayley graph $X(\mathbb{Z}_n, \{1, n - 1\})$ is also called the *n-cycle*. This is the unique connected 2-regular graph up to isomorphism. Let $G \in \mathcal{L}_{6,2}$ be the graph with adjacency matrix

$$A(G) = \rho_{\text{reg}}(s) + \rho_{\text{reg}}(st) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix},$$

where $s = (1, 2)(3, 6)(4, 5)$ and $t = (1, 4)(2, 3)(5, 6)$. Then $s^2 = st^2 = e$ and $\langle s, st \rangle = D_6$. Additionally, it is immediate to see that $G \cong X(\mathbb{Z}_6, \{1, 5\})$.

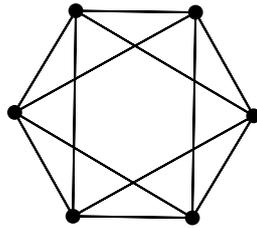


Figure 1. The Cayley graph $X(\mathbb{Z}_6, \{1, 2, 4, 5\})$.

4. Anticirculant weighing matrices

In this section, we will highlight the interplay between weighing matrices and anticirculant graphs supporting orthogonal matrices.

A weighing matrix W of order n and weight k , denoted by $W(n, k)$, is a square $n \times n$ matrix with entries $[W]_{i,j} \in \{-1, 0, 1\}$, satisfying $WW^T = kI$, where I is the identity matrix. When $k = n$, then $W(n, k)$ is said to be a *Hadamard matrix*. It is generally recognized that weighing matrices were first discussed by Frank Yates in 1935 [13], while Hadamard matrices were introduced by James Sylvester and Jacques Hadamard during the second half of the nineteenth century (see [14]). Geramita and Seberry [15] and Koukouvinos and Seberry [16] are general surveys on this topic, as well as applications.

We are specifically interested in circulant weighing matrices. A *circulant weighing matrix* of order n and weight k is a $W(n, k)$ which is also a circulant matrix. In analogy with the previous section, we define the ordered set $T(W(n, k)) = \{s : [A(CW(n, k))]_{i,s} = 1\}$ to be the symbol of $CW(n, k)$. Arasu and Seberry overview the subject in [17].

We shall give particular importance to graphs associated with weighing matrices. The *graph of a* (real symmetric) *matrix* M , denoted by $G(M)$, is the graph defined by

$$[A(G(M))]_{i,j} := \begin{cases} 1, & \text{if } [M]_{i,j} \neq 0; \\ 0, & \text{if } [M]_{i,j} = 0. \end{cases}$$

The graph $G(CW(n, k))$ is a Cayley graph of the cyclic group \mathbb{Z}_n with respect to the set $T(W(n, k))$. For instance, let us look at the matrix

$$CW(6, 4) = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & -1 \\ -1 & 0 & 1 & 1 & 0 & 1 \\ 1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 1 & 0 & 1 & -1 & 0 \end{pmatrix}. \tag{8}$$

The graph $G(CW(6, 4))$ is the Cayley graph of \mathbb{Z}_6 with respect to the set $\{1, 2, 4, 5\}$. This is illustrated in figure 1.

The following statement is one of the main tools in our discussion. For the purposes of this paper, it is sufficient to focus on matrices of even order.

Proposition 3. *For each circulant weighing matrix $M = CW(n, k)$ of order $n = 2l$, with $l \geq 1$, there is a weighing matrix U such that $G(U) \in \mathcal{L}_{n,k}$. The graph $G(U)$ is loopless if each element of $T(W(n, k))$ is odd.*

Proof. Consider the permutation

$$\pi = (1, n), (2, n - 1), \dots, (l, l + 1) \tag{9}$$

on a set $[n]$. Let P_π be the permutation matrix induced by π . Labeling the rows and the columns of P_π with the elements of $[n]$ in the lexicographic order, we can write

$$P_\pi = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & 0 & 1 & 0 \\ 0 & & \ddots & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}.$$

Now let $U = MP_\pi$. Since P_π and M are orthogonal matrices, U is also orthogonal. By the action of the permutation π and the fact that M is circulant, the graph $G(U) \in \mathcal{L}_{n,k}$ and $S(G(U)) = \{s = n - [A(CW(n, k))]_{i,t} + 1\}$, where $t \in T(W(n, k))$. Note that $G(U)$ can have self-loops if we do not impose further restrictions. On the other hand, $G(U)$ is loopless if each $s \in S(G(U))$ is even (proposition 1). Such a property is satisfied only if each element of $T(W(n, k))$ is odd. □

It is important to remark that P_π acts on columns and hence it does not generally preserve isomorphism of graphs. Proposition 3 prompts us to some definitions. An *anticirculant weighing matrix* of order n and weight k , denoted by $AW(n, k)$, is a $W(n, k)$ which is also anticirculant. A matrix $AW(n, k)$ is a special kind of Hankel weighing matrix. A *Hankel weighing matrix* of order n and weight k , denoted by $M = HW(n, k)$, is a $W(n, k)$ such that $G(M)$ is anticirculant. While it is obvious that there exists an $AW(n, k)$ if and only if there exists a $CW(n, k)$, we need to take into account the following counterexample:

Proposition 4. *Not every $HW(n, k)$ is an $AW(n, k)$.*

Proof. Let P_π be the permutation matrix as in the proof of proposition 3. The matrix

$$U = \begin{pmatrix} P_\pi & P_\pi \\ P_\pi & -P_\pi \end{pmatrix} \tag{10}$$

is a $HW(n, 2)$, for every n , since

$$UU^T = U^2 = \begin{pmatrix} 2P_\pi^2 & 0 \\ 0 & 2P_\pi^2 \end{pmatrix} = \begin{pmatrix} 2I & 0 \\ 0 & 2I \end{pmatrix},$$

because $P_\pi = P_\pi^T$. □

Proposition 4 seems to indicate that classification theorems for circulant weighing matrices help only partially when attempting to classify Hankel weighing matrices.

In the following two sections, our classification begins by considering graphs of small degree. This corresponds physically to the number of pump beams required to build a given CVCS. If a graph is r -regular, it requires $2r - 1$ different pump beams to build the associated CVCS, since each of the $2r - 1$ bands of the adjacency matrix requires a different pump frequency. Thus, we are primarily interested in graphs that have an interesting topology, but also have small values of r , since greater values would have greater experimental complexity.

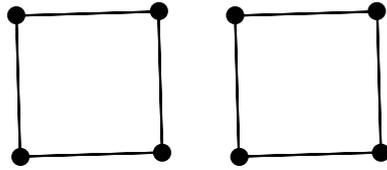


Figure 2. The graph $X(D_8, \{st, st^3\}) \cong C_4 \uplus C_4$.

Cases of small r are also more tractable theoretically, which is another reason to focus on them.

5. 2-regular graphs

The unique—up to isomorphism—connected 2-regular graph on n vertices is the n -cycle, C_n . A graph G is said to be the *disjoint union* of graphs H_1, \dots, H_l if there is a permutation matrix P such that $PA(G)P^{-1} = \bigoplus_{i=1}^l A(H_i)$. In such a case, we write $G \cong \bigsqcup_{i=1}^l H_i$. Equivalently, let G and H be graph such that $V(G) \cap V(H) = \emptyset$. The set of vertices of $G \uplus H$ is $V(G) \cup V(H)$ and the set of edges, $E(G) \cup E(H)$. All useful information about graphs of matrices $HW(n, 2)$ is stated as follows:

Proposition 5. *Let D_n be the dihedral group of order $n = 2k$. Let $T = \{st^i, st^j\} \subset D_n$, where $1 \leq i, j \leq k$, be a set of reflections. There is a unitary matrix U such that $X(D_n, T) \cong G(U)$ if and only if $(i - j) \bmod k = (j - i) \bmod k$. Moreover, if this is the case then k is even and therefore n is a multiple of 4. It follows that the graph $G \in \mathcal{L}_{n,2}$ and G is the disjoint union of $k/2$ copies of C_4 .*

Proof. It is known that given a group Γ and a set $T = \{g, h\} \subset \Gamma$, there is a unitary matrix U such that $A(X(\Gamma, T)) = A(G(U))$ if and only if $gh^{-1} = g^{-1}h$ (see, e.g., [18, 19]). From this, $st^i(st^j)^{-1} = (st^i)^{-1}st^j$. Since st^l is a reflection, for every $1 \leq l \leq k$, we can write $(st^l)^{-1} = st^l$. Thus $st^i st^j = st^j st^i$, that is, st^i and st^j commute. Since $st^l s = t^{k-l}$, for every $1 \leq l \leq k$, it follows that $t^{(k-i+j) \bmod k} = t^{(k-j+i) \bmod k}$. This proves that $(i - j) \bmod k = (j - i) \bmod k$, for st^i and st^j . Now, without loss of generality, assume that $j > i$. Note that $(i - j) \bmod k = k - j + i$. So, we need that $j - i = k - j + i$, which implies $j - i = k/2$. It follows that U has the form described in equation (10), up to isomorphism. \square

Figure 2 illustrates the graph $X(D_8, \{st, st^3\})$ associated with the matrix $HW(8, 2)$. Since in general only a disjoint union of squares can be achieved with 2-regular graphs, and these graphs have only limited interest from a quantum computing perspective, we need to consider larger graphs. A proposal for creating such graphs was given in [9].

6. 4-regular graphs

In this section, we consider several constructions of 4-regular graphs of anticirculant weighing matrices. Unfortunately, we are unable to prove that this set of examples is exhaustive, so a complete characterization of the degree 4 case remains open.

The CVCS graphs in this section can all be implemented with an experimental setup requiring no more than seven pump beams.

6.1. Graphs from $AW(4, 4)$

The matrix

$$M_{4,4} = AW(4, 4) = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

is a Hadamard matrix of order 4. Since the entries of a Hadamard matrix are all nonzero, $G(M_{4,4}) \cong K_4^+$, where K_n^+ denotes the complete graph on n vertices with a self-loop at each vertex. We can construct other anticirculant weighing matrices of higher order from $M_{4,4}$. Given that we are interested in anticirculant matrices whose graph is without self-loops, the first row of these matrices must have nonzero entries only at even positions. Let $\mathcal{M}_{4,4}$ be a family of matrices obtained from $M_{4,4}$ by adding new lines (i.e., rows and columns) between the rows and the columns of $M_{4,4}$. If we add the same number of lines between each column (resp. row), we obtain matrices of order $4k$, for $k \geq 1$. For small orders, the first rows of these matrices are

$$\begin{aligned} &0, 1, 0, 1, 0, 1, 0, -1 \\ &0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, -1 \\ &0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, -1 \\ &0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, -1. \end{aligned}$$

Clearly, for each matrix M constructed in this way, we have $G(M) \in L_n$. When $n = 4k$, with k odd, the graph $G(M)$ has self-loops. For this reason, we consider only the case $n = 4k$, with k even. In other words, we consider the case $n = 8l$, for any $l \geq 1$. Let $S(M) = \{s_1, s_2, s_3, s_4\}$. By the above construction $s_{i+1} - s_i = d + 1$ for $i = 1, 2, 3$ and $n - s_4 + 2s_1 = s_2$. This last condition provides that the number of columns between s_4 and s_1 is the same as one between s_2 and s_1 when looking at the matrix as wrapped on a torus, by gluing together the first and the last column. In this way, we can define a unitary matrix U such that $G(U) \cong G(M)$. Let us denote by U_1, \dots, U_n the rows of U . The nonzero entries in U_1 are exactly $U_{1,s_1}, U_{1,s_2}, U_{1,s_3}$ and U_{1,s_4} . In particular, $U_{1,s_1} = U_{1,s_2} = U_{1,s_3} = 1$ and $U_{1,s_4} = -1$. Suppose $s_{i+1} - s_i - 1 = d$ for all $i = 1, 2, 3$ and $n - s_4 + 2s_1 = s_2$. Thus the inner product between the rows U_1 and $U_{s_2-s_1}$ is

$$\begin{aligned} \langle U_1, U_{s_2-s_1} \rangle &= U_{1,s_1}U_{l,s_1} + U_{1,s_2}U_{l,s_2} + U_{1,s_3}U_{l,s_3} + U_{1,s_4}U_{l,s_4} \\ &= U_{1,s_1}U_{1,s_2} + U_{1,s_2}U_{1,s_3} + U_{1,s_3}U_{1,s_4} + U_{1,s_4}U_{1,s_1} \\ &= 1 + 1 - 1 - 1 \\ &= 0. \end{aligned}$$

The distance d guarantees that each row (resp. column) of U has exactly four nonzero entries contributing to the zero inner product with other three rows (resp. columns). Orthogonality is guaranteed since no nonzero entries contribute to the inner product with all remaining rows (resp. columns). The numbers s_1, \dots, s_4 must be even and $d = (n - 4)/4$ must be odd, since $n = 8l = (4d + 4)$. The matrices of order $8l$ constructed in with this method will be denoted by $M_{4,4,l}$. Essentially, orthogonality arises since we interlace $M_{4,4}$ with itself $d + 1$ times.

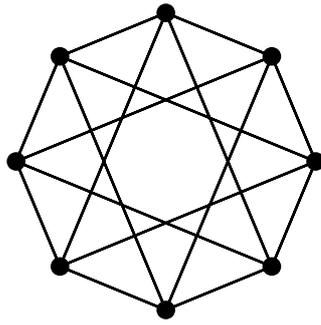


Figure 3. The graph $G(M_{4,4,1}) \cong K_{4,4}$.

This can be seen directly in the matrix

$$M_{4,4,1} = AW(8, 4) = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & -1 \\ \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{-1} & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\ \boxed{1} & 0 & \boxed{1} & 0 & \boxed{-1} & 0 & \boxed{1} & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \\ \boxed{1} & 0 & \boxed{-1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 \\ 0 & -1 & 0 & 1 & 0 & 1 & 0 & 1 \\ \boxed{-1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 & \boxed{1} & 0 \end{pmatrix}, \quad (11)$$

which is constructed by interlacing two copies of $M_{4,4}$. The boxed numbers are the nonzero entries in one of the two copies. At this stage, we can ask information about the graphs of matrices in $\mathcal{M}_{4,4}$. It is sufficient to observe that $M_{4,4,l} = M_{4,4} \otimes F_{l+1}$, where the matrix

$$F_n = \begin{pmatrix} 0 & \dots & 0 & 1 \\ \vdots & & 1 & 0 \\ 0 & & & \vdots \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

is $n \times n$. Note that this is just the matrix P_π from proposition 3, but we have changed notation to make the dependence on the size of the matrix explicit. The matrix F_n is the adjacency matrix of the disjoint union of $n/2$ graphs K_2 , if n is even and the disjoint union of $(n - 1)/2$ graphs K_2 and a single vertex with a self-loop, if n is odd. The *tensor product* $G \otimes H$ of graphs G and H is the graph such that $A(G \otimes H) = A(G) \otimes A(H)$. The set of vertices of $G \otimes H$ is the Cartesian product $V(G) \times V(H)$ and two vertices $\{u, u'\}$ and $\{v, v'\}$ are adjacent in $G \otimes H$ if and only if $\{u, v\} \in E(G)$ and $\{u', v'\} \in E(H)$. The graph $G \otimes H$ is connected if and only if both factors are connected and at least one factor is nonbipartite. Here, K_2 and K_4 are both connected and K_2 is bipartite. When $l > 1$, one of the factors is not connected. Therefore $G(M_{4,4,l})$ is connected only if $l = 1$. In fact $G(M_{4,4,l}) \cong \bigoplus_{\#l} K_{4,4}$, where $K_{n,n}$ denotes the complete bipartite graph on $2n$ vertices. In figure 3 is illustrated $G(M_{4,4,1})$; in figure 4, $G(M_{4,4,4})$.

The graph $G(M_{4,4,l})$ is made of l connected components. Can we find two permutation matrices P and Q such that $G(PM_{4,4,l}Q) \cong G(M_{4,4,1})$ and it is connected? In particular, for the moment, we consider only permutation matrices that preserves the distance d between the

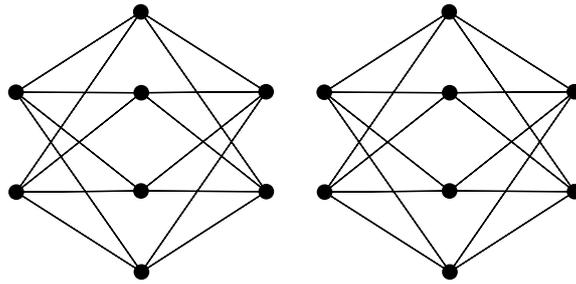


Figure 4. The graph $G(M_{4,4,4}) \cong K_{4,4} \uplus K_{4,4}$.

elements of $S(M)$. From $M_{4,4,l}$, we can construct exactly $l - 1$ further matrices, by permutation the lines under this constraint. For example, when $n = 24 = 8 \times 3$, the possible first rows are

$$\begin{aligned} &0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1 \\ &0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0. \\ &0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, -1, 0, 0, 0, 0 \end{aligned}$$

The position of the entry -1 does not matter, because we are interested in the graph $G(M_{4,4,l})$. Given s_1, \dots, s_4 , in matrices of the form $M_{4,4,l}$, when $l > 1$,

$$\gcd(\{s_i - s_j : \text{for all } i, j \text{ such that } j > i\} \cup 8l) > 2.$$

This implies that $G(PM_{4,4,l}Q) \cong \bigsqcup_{\#l} K_{4,4}$, for any two permutations P and Q preserving the distance d .

6.2. Graphs from $AW(6, 4)$

It is natural to analyze the construction in the previous section, but replacing the matrix $AW(4, 4)$ with some other anticirculant weighing matrix. Recalling that there is no $AW(5, 4)$, the smallest available matrix of order $n > 4$ is $AW(6, 4)$,

$$M_{6,4} = AW(6, 4) = \begin{pmatrix} -1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

This is obtained from the $CW(6, 4)$ in (8). The graph $G(M_{6,4})$, illustrated in figure 5, has four self-loops.

The graph $G(M_{6,4,l})$, if loopless, is on $12l$ vertices, with $l \geq 1$. The graph $G(M_{6,4,1}) \cong G(M_{6,4}) \otimes K_2$ is connected since K_2 and $G(M_{6,4})$ are connected and $G(M_{6,4})$ is nonbipartite. (The chromatic number of $G(M_{6,4})$ is 3.) A drawing of $G(M_{6,4,1})$ is in figure 6.

In general, $M_{6,4,l} = M_{6,4} \otimes F_{l+1}$. If we avoid self-loops,

$$G(M_{6,4,l}) = \bigsqcup_l G(M_{6,4,1}).$$

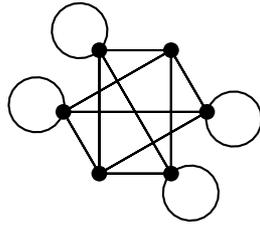


Figure 5. The graph $G(M_{6,4})$.

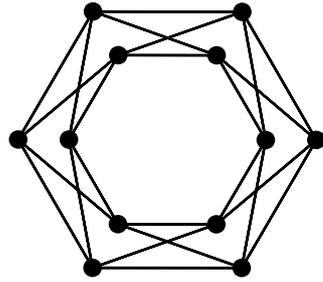


Figure 6. The graph $G(M_{6,4,1}) \cong G(M_{6,4}) \otimes K_2$.

By permuting the lines of $M_{6,4}$ (*shifting to the right the columns*), we have the matrix

$$M_{6,4}^1 = AW(4, 4)_1 = \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & -1 & 1 \\ 1 & 1 & 0 & -1 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 1 \end{pmatrix}.$$

Apply our construction to $M_{6,4}^1$, we obtain graphs on $6k$ vertices, with k odd. However, since $G(M_{6,4,1}) \cong G(M_{6,4,1}^1)$, these graphs will be a disjoint union of copies of $G(M_{6,4,1})$. So the permutation is of no use if our intention is to get different graphs from $M_{6,4}$. On the other side, the structure of $M_{6,4}^1$ suggests a different construction, which will allow us to obtain matrices of the form $AW(4m, 4)$, whose graph is loopless and connected for every $m \geq 3$. Note that

$$M_{6,4} = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Then $2AB = 0$ and $A^2 + B^2 = 4I$ and indeed

$$M_{6,4}M_{6,4} = (M_{6,4})^2 = \begin{pmatrix} A^2 + B^2 & 2AB \\ 2AB & A^2 + B^2 \end{pmatrix} = 4I.$$

We can add an even number of extra lines to $M_{6,4}$ and still preserve its block structure. For example, the matrices

$$A' = \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}$$

satisfy the above conditions. Hence the matrix

$$M_{12,4} = \begin{pmatrix} A' & B' \\ B' & A' \end{pmatrix}$$

is an $AW(12, 4)$. In general, let $\mathcal{M}_{6,4}$ be the family of matrices obtained with this construction. If $M \in \mathcal{M}_{6,4}$ then $s_1 = 2, s_2 = 4, s_3 = n/2+2$ and $s_4 = n/2+4$, where $S(M) = \{s_1, s_2, s_3, s_4\}$. These are $AW(4k, 4)$ when $s_3 - s_2 = (n - 6) / 2$ is an odd number. The construction gives also matrices of order $2l \geq 6$, for any l . Yet the associated graphs will have loops when l is odd. This is the reason for taking multiples of 4. If $M \in \mathcal{M}_{6,4}$ then $G(M)$ is connected, in virtue of the fact that $s_2 - s_1 = 4 - 2 = 2$ and then $\gcd(\{s_i - s_j : \text{for all } i, j \text{ such that } j > i\} \cup 4k) = 2$.

The graphs of matrices in the set $\mathcal{M}_{6,4}$, with order $4m$, are the direct product of C_{2m} with the graph K_2^+ . (Recall that K_n^+ is the complete graph on n vertices with self-loops; these are the graphs with adjacency matrix J_n .) The *direct product* $G \times H$ of graphs G and H has a set of vertices $V(G \times H) = V(G) \times V(H)$ and two vertices $\{u, u'\}$ and $\{v, v'\}$ are adjacent in $G \times H$ when $\{u, v\} \in E(G)$ and $\{u', v'\} \in E(H)$. By the definition, $A(G \times H) = A(G) \otimes A(H)$. For this reason $G \times H$ is sometimes called the *Kronecker product of graphs* (see Imrich and Klavžar [20], chapter 5). If $G \in \mathcal{M}_{6,4}$ then $G \cong K_2^+ \times C_{2m}$ and $M(G) = J_2 \otimes A(C_{2m})$.

6.3. Graphs from $AW(7, 4)$

Let us consider the matrix

$$M_{7,4} = AW(7, 4) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 1 \\ 1 & 1 & -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{pmatrix}.$$

The graph of $M_{7,4}$ is in figure 7.

Interlacing $M_{7,4}$ with itself, by the same method used for $M_{4,4}$ (see section 6.1, equation (11)), we obtain the matrix

$$M_{7,4,1} = AW(14, 4) = AW(7, 4) \otimes F_2. \tag{12}$$

The graph of $M_{7,4,1}$ is in figure 8.

This is on 14 vertices, 4-regular and connected. At a first analysis, $G(M_{7,4,1})$ does not seem to be a direct product of graphs. It is nontrivial to extend $G(M_{7,4,1})$ to an infinite family of connected regular graphs, as we did by taking $M_{6,4}$. Examples show that graphs obtained by $M_{7,4}$ are not connected, but $G(M_{7,4,1})$.

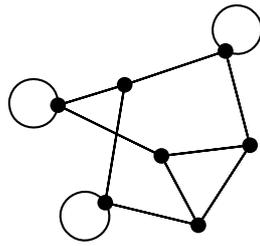


Figure 7. The graph $G(M_{7,4})$.

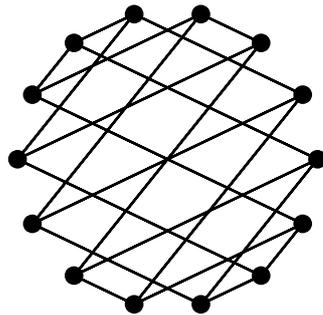


Figure 8. The graph $G(M_{7,4,1})$.

7. Open problems

- Study the structure of the graph $G(M_{7,4,1})$. Determine if it belongs to a family of connected graphs arising from $AW(7, 4)$, having an infinite number of members, as it is for the graphs arising from $AW(6, 4)$.
- Since there is no general classification of weighing matrices, characterizing graphs of Hankel weighing matrices seems to be out of reach. This could be however done for graphs of degree 4 and 9, on the basis of present knowledge on $CW(n, 4)$ and $CW(n, 9)$ (see [17]).
- We have seen that there are examples of Hankel weighing matrices which are not anticirculant. A way to study the relation between these matrices and anticirculant ones, would be to prove that the graphs of Hankel weighing matrices are always isomorphic to graphs of $AW(n, d)$, except for some special cases which will include the examples in proposition 4.
- A *Hamiltonian cycle* in a graph is an ordered set of sequentially adjacent vertices, in which every vertex of the graph appears exactly once. Graphs with Hamiltonian cycles are said to be *Hamiltonian*. Alspach and Zhang [21] conjectured that a connected Cayley graph $G(D_n, T)$ is Hamiltonian whenever T is a set of reflections. According to this conjecture, any connected anticirculant graph is Hamiltonian whenever it is loopless. Additionally, Gutin *et al* [22] conjectured that graphs of unitary matrices are Hamiltonian if connected. According to these conjectures the graphs of Hankel weighing matrices are Hamiltonian if connected. This is the case for the graphs considered here. In figure 9 the graphs $K_2^+ \times C_8$ and $G(M_{7,4,1})$ are drawn. The highlighted edges represent Hamiltonian cycles.

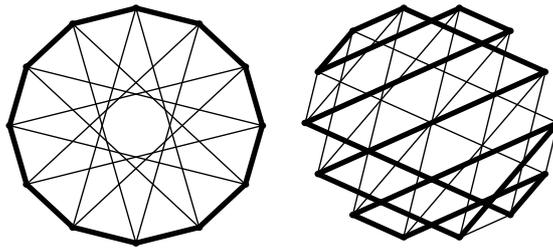


Figure 9. Hamiltonian cycles in graphs of matrices $AW(12, 4)$ and $AW(14, 4)$.

Perhaps the physical picture that motivates our study could provide some insight toward proving these conjectures.

- We have seen that graphs of the form $K_2^+ \times C_{2m}$ are graphs of unitary matrices. The quantum dynamics induced by these matrices is trivial because of symmetry. However, symmetry can be broken with the action of a diagonal matrix without altering the zero pattern of the unitary. This suggests the possibility of constructing unitaries with a nontrivial dynamics whose graph is a undirected Cayley graph of the dihedral group.
- Sections 5 and 6 make extensive use of the following trick. Begin with a circulant weighing matrix, then reverse the columns. If the resulting graph has loops, one can create a loopless graph by the ‘interlacing’ trick. In fact the interlaced graph will again be connected if the original graph was connected but not bipartite. Can one classify the Hankel and anticirculant weighing matrices which cannot be obtained by this trick?

8. Conclusion

We have demonstrated connections between an approach to quantum computation with optical modes and the existence of certain weighing matrices with Hankel structure. Because the degree of the graph associated with a matrix corresponds to the experimental resources required to implement the graph as a CVCS, it is important to characterize which graphs with small degree have adjacency matrices with Hankel structure. We proved some general theorems about such matrices, classified completely the case of degree 2, and provided some examples with degree 4. We also raised some interesting open problems that might help shed some light on which graphs can be implemented in a single OPO using the scheme in [7, 10].

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References

- [1] Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

- [2] Raussendorf R and Briegel H J 2001 A one-way quantum computer *Phys. Rev. Lett.* **86** 5188–91
- [3] Briegel H J and Raussendorf R 2001 Persistent entanglement in arrays of interacting particles *Phys. Rev. Lett.* **86** 910–3
- [4] Zhou D L, Zeng B, Xu Z and P. C 2003 Sun quantum computation based on d-level cluster states *Phys. Rev. A* **68** 062303
- [5] Zhang J and Braunstein S L 2006 Continuous-variable Gaussian analog of cluster states *Phys. Rev. A* **73** 32318
- [6] Menicucci N C, Loock P van, Gu M, Weedbrook C, Ralph T C and Nielsen M A 2006 Universal quantum computation with continuous-variable cluster states *Phys. Rev. Lett.* **97** 110501
- [7] Menicucci N C, Flammia S T and Pfister O 2008 One-way quantum computing in the optical frequency comb *Phys. Rev. Lett.* **101** 130501
- [8] Menicucci N C, Flammia S T, Zaidi H and Pfister O 2007 Ultracompact generation of continuous-variable cluster states *Phys. Rev. A* **76** 010302
- [9] Zaidi H, Menicucci N C, Flammia S T, Bloomer R, Pysker M and Pfister O 2008 Entangling the optical frequency comb: simultaneous generation of multiple 2×2 and 2×3 continuous-variable cluster states in a single optical parametric oscillator *Laser Phys.* **18** 659
- [10] Flammia S T, Menicucci N C and Pfister O 2008 The optical frequency comb as a one-way quantum computer arXiv:0811.2799
- [11] Diestel R 2000 *Graph Theory (Graduate Texts in Mathematics)* 2nd edn (New York: Springer)
- [12] Cameron P J 1999 *Permutation groups (London Mathematical Society Student Texts vol 45)* (Cambridge: Cambridge University Press)
- [13] Yates F 1935 Complex experiments *J. R. Stat. Soc. (Suppl.)* **2** 181–247
- [14] Horadam K J 2007 *Hadamard Matrices and Their Applications* (Princeton, NJ: Princeton University Press)
- [15] Geramita A V and Seberry J 1979 *Orthogonal Designs, Quadratic Forms and Hadamard Matrices (Lecture Notes in Pure and Applied Mathematics vol 45)* (New York: Dekker)
- [16] Koukouvinos C and Seberry J 1997 Weighing matrices and their applications *J. Stat. Plan. Inference* **62** 91–101
- [17] Arasu K T and Seberry J 1998 On circulant weighing matrices *Australas. J. Combin.* **17** 21–37
- [18] Severini S 2003 On the digraph of a unitary matrix *SIAM J. Matrix Anal. Appl. (SIMAX)* **25** 295–300
- [19] Severini S 2008 Graphs of unitary matrices *Ars Combin.* **89**
- [20] Imrich W and Klavžar S 2000 *Product Graphs. Structure and Recognition Wiley-Interscience Series in Discrete Mathematics and Optimization* (New York: Interscience)
- [21] Alspach B and Zhang C-Q 1989 Hamilton cycles in cubic Cayley graphs on dihedral groups *Ars Combin.* **28** 101–8
- [22] Gutin G, Rafiey A, Severini S and Yeo A 2006 Hamilton cycles in digraphs of unitary matrices *Discrete Appl. Math.* **173** 67–78