A conjecture on isomorphism of strongly regular graphs [1]

A d-regular graph $G = (V, E)$ on n vertices is called a *strongly regular graph*, and it is denoted by $srg(n, d, l, m)$, if the following conditions are satisfied:

1. any two vertices i and j, such that $\{i, j\} \in E(G)$, are both adjacent to a constant number l of vertices (independent of the choice of the adjacent pair $\{i, j\}$);
2. any two distinct vertices i and j, such that $\{i, j\} \notin E(G)$, are both adjacent to a constant number m of vertices (independent of the choice of the nonadjacent pair $\{i, j\}$).

The adjacency matrix $A(G)$ satisfies

$$A(G)J = dJ \quad \text{and} \quad A^2(G) + (m - l)A(G) + (m - d)I = mJ,$$

where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ all-ones matrix.

Let us write

$$A(G) = \sum_{i=1}^{d} P_i,$$

where $P_1, P_2, ..., P_d$ are permutation matrices. We define the *Grover walk matrix* as

$$U(G) = \bigoplus_{i=1}^{d} P_i \cdot (C_d \otimes I),$$

where

$$C_d = I_d - \frac{2}{d} J_d.$$

Notice that the matrix $U(G)$ is real-orthogonal, *i.e.*, $U^T(G)U(G) = U(G)U^T(G) = I_{dn}$.

Given an $n \times n$ real matrix M, let us denote by M^+ the $n \times n$ matrix defined as follows:

$$M^+_{i,j} = \begin{cases} 1, & \text{if } M_{i,j} > 0; \\ 0, & \text{otherwise.} \end{cases}$$

Recall that two graphs $G = (V, E)$ and $H = (W, F)$ are said to be *isomorphic*, and in such a case we write $G \cong H$, whenever there is a bijection $f : V(G) \longrightarrow V(H)$ such that $\{i, j\} \in E(G)$ if and only if $\{f(i), f(j)\} \in E(H)$, for every $i, j \in V(G)$.

Equivalently, G and H are isomorphic if and only if there is a permutation matrix P such that

$$PA(G)P^T = A(H).$$

The *spectrum* of a matrix M, denoted by S_M, is the multiset of the eigenvalues of M. Clearly, if $G \cong H$ then

$$S_{A(G)} = S_{A(H)}.$$

We ask the following question:

Problem 1 *Let G and H be two $srg(n, d, l, m)$. Then, $G \cong H$ if and only if*

$$S_{(U^3(G))^+} = S_{(U^3(H))^+}.$$

[1] D. Emms, S. Severini, R. Wilson, E. Hancock, A matrix representation of graphs and its spectrum as a graph invariant, The Electronic Journal of Combinatorics, R34, Volume 13(1), 2006. arXiv:quant-ph/0505026v2 [See also K. J. Guo, Quantum Walks on Strongly Regular Graphs, Master Dissertation, University of Waterloo, 2010.]