The Rank-1 Quantum Chromatic Number

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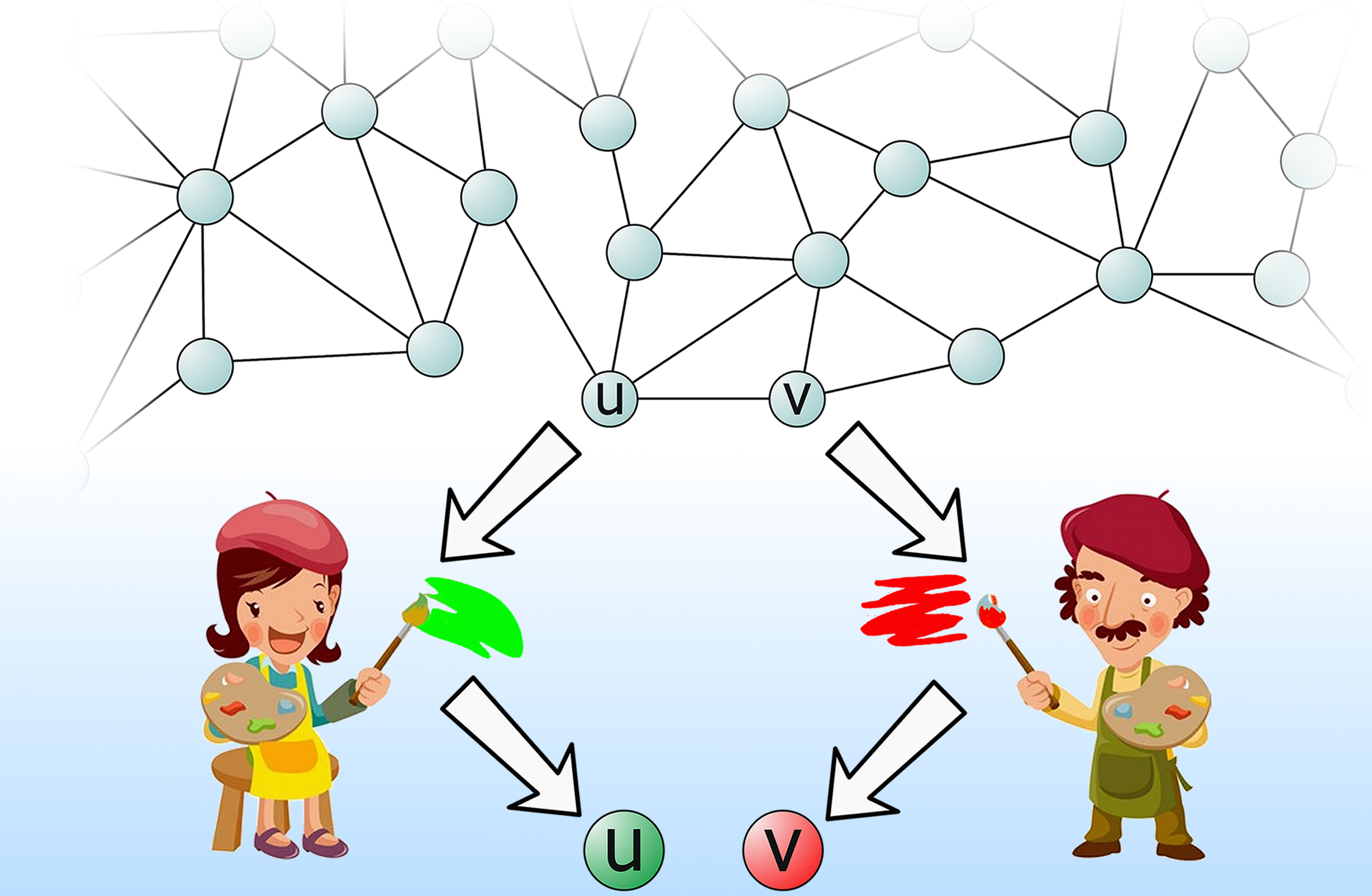

Introduction

We study the relationship between the rank-1 quantum chromatic number and other graph parameters: the orthogonal rank, the chromatic number and the zero-error entanglement-assisted channel capacity.

The Coloring Game

Let G be a graph. Alice gets a vertex u and Bob gets a vertex v from $V(G)$. They output colors $c(u)$ and $c(v)$ respectively, from a set of k allowed colors. They win the game if the following holds:

- $u = v \Rightarrow c(u) = c(v)$
- $(u, v) \in E(G) \Rightarrow \alpha \neq \beta$

Classically, the minimum k such that the players win the coloring game with certainty is the chromatic number $\chi(G)$. If the players share entanglement, the minimum k is called *quantum chromatic number* $\chi_q(G)$.

The **rank-1 quantum chromatic number**, $\chi_q^{(1)}(G)$, is an upper bound $\chi_q(G)$, obtained by restricting player's measurements to be rank-1 projectors. It turns out that $\chi_q^{(1)}(G)$ is the minimum k such that there exist orthonormal bases $\{|a_{vi}\rangle\}_{i\in[k]}$ for all $v \in V(G)$, satisfying:

- $\langle a_{vi}, a_{vj}\rangle = 0, \forall i \neq j, \forall v \in V(G)$
- $\langle a_{ui}, a_{vi}\rangle = 0, \forall i, \forall (u, v) \in E(G)$.

Other Definitions

Orthogonal rank $\xi(G)$: the minimum d such that there exists a orthogonal representation of G in d dimensions (i.e. a function mapping adjacent vertices to orthogonal vectors in $\mathbb{C}^d$).

Kochen-Specker (KS) set: a set $S \subseteq \mathbb{C}^n$ s.t. there is no *marking function* $f : \mathbb{C}^n \to \{0, 1\}$ satisfying that for all orthonormal bases $b \subseteq S$, $\sum_{u\in b} f(u) = 1$ (exactly one element is marked).

Weak KS set: a set $S \subseteq \mathbb{C}^n$ s.t. for all *marking functions* defined as above, there exist orthogonal vectors $u, v \in S$ for which $f(u) = f(v) = 1$.

Main Results

For all graphs G, we have the relation

$$\xi(G) \leq \chi_q^{(1)}(G) \leq \chi(G).$$

Using Kochen-Specker sets, we show that the first inequality is strict and we give a necessary and sufficient condition for the second inequality to be strict.

Theorem 1. *There are graphs s.t.* $\xi(G) < \chi_q^{(1)}(G)$.

Proof sketch. Let G_S be an orthogonality graph of a KS set $S \subseteq \mathbb{C}^3$. We have $\xi(G_S) = 3$, and we claim that $\chi(G_S) > 3$. Suppose we are able to 3-color G_S, then we can build the marking function

$$f(v) = 1 \Leftrightarrow c(v) = i.$$

Every orthonormal basis $b \subseteq S$ is a clique in G_S, so we have $\sum_{u\in b} f(u) = 1$, contradicting the assumption that S is a KS set.

In [1] it is proven that $\chi_q^{(1)} = 3$ if and only if $\chi = 3$, so we conclude that $\chi_q^{(1)}(G_S) > 3$. □

Theorem 2. *For all graphs,* $\chi_q^{(1)}(G) < \chi(G)$ *if and only if for all optimal rank-1 quantum colorings*

$$S = \{|a_{vi}\rangle : v \in V, i \in [k]\}$$

is a weak Kochen-Specker set.

Proof sketch. $\Rightarrow$ Let $k = \chi_q^{(1)}(G) < \chi(G)$. If S is *not* a weak KS set, there is a marking function f that: marks one basis vector $|a_{vi}\rangle$ per vertex and for edges (u, v), $f(|a_{ui}\rangle) \neq f(|a_{vi}\rangle)$. Therefore there is the following classical k-coloring:

$$c(v) = i \Leftrightarrow f(|a_{vi}\rangle) = 1.$$

$\Leftarrow$ Let $\chi_q^{(1)}(G) = k$ and assume that for all optimal rank-1 colorings, S is a weak KS set. Now suppose that also $\chi(G) = k$. Then for each $v \in V$ with color j, define the rank-1 quantum coloring

$$|a_{vi}\rangle = |i + j\rangle.$$

The set S is just the standard basis of $\mathbb{C}^k$, therefore not a weak KS set, contradicting the assumption. □

Channel Capacity

We propose a new family of channels with *one-shot zero-error entanglement-assisted capacity*, c_0^*, larger than the *one-shot zero-error capacity*, c_0.

Let G be a graph on n vertices with $\chi_q^{(1)}(G) = k$, we define a channel $\mathcal{N}_G$ as follows:

- Confusability graph $G \Box K_k$ (Cartesian product of G and the complete graph on k vertices)
- Unique output symbol $y_{u,v}$ for every pair of confusable inputs (u, v).

Theorem 3. *Let G be a graph with n vertices and* $\chi(G) > \chi_q^{(1)}(G) = k$. *Then* $c_0(\mathcal{N}_G) < n \leq c_0^*(\mathcal{N}_G)$. *Moreover,* $c_0(\mathcal{N}_G) \leq \alpha(G) \cdot k$.

Where $\alpha(G)$ is the independence number of G. We prove the theorem by combining together the properties of weak KS sets, a protocol by Cubitt *et al.* [3] and a classic theorem by Vizing.

Also, using this theorem and a known bound by Frankl and Rödl, we have the following corollary.

Corollary 1. *For all* Hadamard graphs Ω_n, *there is an exponential gap between* $c_0^*(\mathcal{N}_{\Omega_n})$ *and* $c_0(\mathcal{N}_{\Omega_n})$.

Conclusions

We have shown the connection between Kochen-Specker sets and the graph coloring games.

An **open question** is to find a “converse” for Theorem 2. Is there a way, starting from a weak Kochen-Specker set, to exhibit a graph with a separation between χ and $\chi_q^{(1)}$?

Main References

[1] Cameron, Montanaro, Newman, Severini and Winter. "On the Quantum Chromatic Number of a Graph". Electronic Journal of Combinatorics, 14(1), 2007.

[2] Scarpa and Severini. "Kochen-Specker sets and the Rank-1 Quantum Chromatic Number". To appear in IEEE Transactions on Information Theory. arXiv:1106.0712.

[3] Cubitt, Leung, Mattews, Winter. "Improving Zero-Error classical communication with entanglement". Phys. Rev. Lett. 104(23), 2010.

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