On zero-error communication via quantum channels in the presence of noiseless feedback

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We initiate the study of zero-error capacity of quantum channels when the receiver and sender have at their disposal a noiseless feedback channel of unlimited quantum capacity, in analogy to Shannon's zero-error communication theory with instantaneous feedback. We first show that this capacity is a function only of the linear span of Kraus operators of the channel, which generalizes the bipartite equivocation graph in the classical case. This then opens the door to the search for a quantum analogue of the fractional packing number of a bipartite graph. We present a candidate for this, motivated by a conjecture originally made my Shannon and proved by Ahlswede, which we demonstrate to have many desirable properties, including being additive and an upper bound to the feedback-assisted zero-error capacity. We also sketch a coding theorem for a weaker version of this capacity attaining the upper bound, but have to leave open the question whether the (strict) feedback-assisted zero-error capacity – plus a constant amount (i.e. zero rate) of forward communication – and our upper bound always coincide. We illustrate our ideas with a number of examples: Weyl-diagonal channels and classical-quantum channels.

I. Zero-error communication assisted by feedback. Consider a quantum channel $\mathcal{N} : \mathcal{S}(A) \longrightarrow \mathcal{S}(B)$, i.e. a completely positive and trace preserving (cptp) linear map from the operators on A to those of B (both finite-dimensional Hilbert spaces), with Kraus and Stinespring representations

$$\mathcal{N}(\rho) = \sum_j E_j \rho E_j^\dagger = \mathrm{Tr}_C\, V\rho V^\dagger,$$

for linear operators $E_j : A \longrightarrow B$ such that $\sum_j E_j^\dagger E_j = \mathbb{1}$, and an isometry $V : A \longrightarrow B \otimes C$, respectively. We will discuss a model of communication where Alice uses the channel n times in succession, allowing Bob after each round to send her back an arbitrary quantum system. They may also share an entangled state prior to the first round. Their goal is to allow Alice to send one of M messages down the channel uses such that Bob is able to distinguish them perfectly. More formally, the most general *quantum feedback-assisted code* consists of a state (w.l.o.g. pure) $|\phi\rangle \in X_0 \otimes Y_0$ and for each message $m = 1, \ldots, M$ isometries for encoding and feedback decoding

$$U_t^{(m)} : X_{t-1} \otimes F_{t-1} \longrightarrow A_t \otimes X_t,$$
$$W_t : Y_{t-1} \otimes B_t \longrightarrow F_t \otimes Y_t,$$

for $t = 1, \ldots, n$ and appropriate local quantum systems X_0 (Alice) and Y_t (Bob), as well the feedback-carrying systems F_t; see Fig. 1. For consistency (and w.l.o.g.), $F_0 = F_n = \mathbb{C}$ are trivial.

We call it a *zero-error code* if there is a measurement on Y_n that distinguishes Bob's output states $\rho_m = \sum_{\underline{j}} \rho_m^{(\underline{j})}$, with certainty, where the sum is over the states

$$\rho_m^{(\underline{j})} = \mathrm{Tr}_{X_n}\left(\prod_{t=n}^{1} (W_t E_{j_t} U_t^{(m)}) |\phi\rangle\langle\phi| \prod_{t=1}^{n} ({U_t^{(m)}}^\dagger E_{j_t}^\dagger W_t^\dagger) \right),$$

the output states given a specific sequence $\underline{j} = j_1 \ldots j_n$ of Kraus operators. In other words, these states ρ_m have to have mutually orthogonal supports, i.e. for all $m \neq m'$, all $\underline{j}$, $\underline{k}$ and $\xi \geq 0$ on $\mathcal{L}(X_n)$,

$$0 = \langle\phi| \prod_{t=1}^{n} ({U_t^{(m')}}^\dagger E_{j_t}^\dagger W_t^\dagger) \xi \prod_{t=n}^{1} (W_t E_{j_t} U_t^{(m)}) |\phi\rangle.$$

By linearity, we see that this condition depends only on the linear span of the Kraus operators,

$$K = \mathcal{K}(\mathcal{N}) = \mathrm{span}\{E_j : j\},$$

in fact it can evidently be expressed as the orthogonality of a tensor defined as a function of $|\phi\rangle$, the $U_t^{(m)}$ and W_t to the subspace $(K \otimes K^\dagger)^{\otimes n}$ – cf. similar albeit simpler characterisations of zero-error and entanglement-assisted zero-error codes in terms of $S = K^\dagger K$ [7].

Proposition 1 *A feedback-assisted code for a channel $\mathcal{N}$ being zero-error is a property solely of the Kraus space $K = \mathcal{K}(\mathcal{N})$. In particular, the feedback-assisted zero-error capacity of $\mathcal{N}$, $\lim_n \frac{1}{n} \log \max M$, is a function only of K, and will be denoted as $C_{0EF}(K)$.*

In the case of a classical channel $N : \mathcal{X} \longrightarrow \mathcal{Y}$, with transition probabilities $N(y|x)$, assisted by classical noiseless feedback, the above problem was first studied – and completely solved – by Shannon [9]. Shannon proved that, if also a finite *amount*, i.e. zero-rate, of

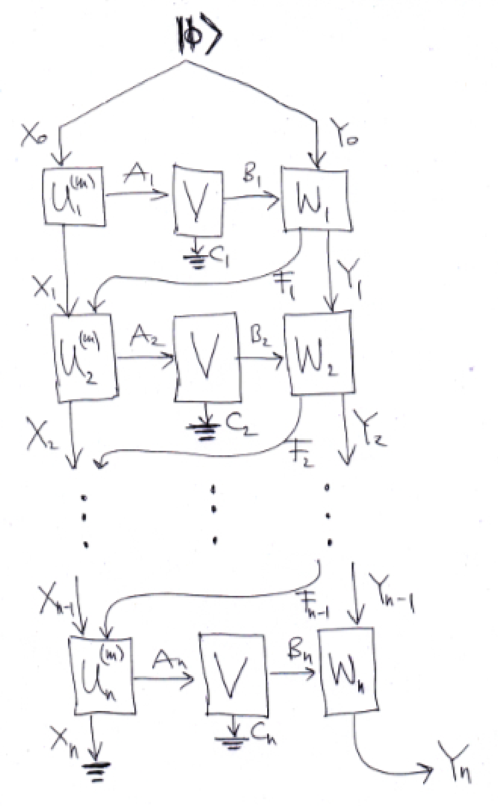

FIG. 1. Diagrammatic representation of a feedback-assisted code for messages m sent down a channel $\mathcal{N}$ used n times, in the form of a schematic circuit diagram. All boxes are isometries (acting on suitably large input and output quantum registers), and the solid lines and arrows represent the "sending" of the respective register. Bob's final output state ρ_m after n rounds of using the channel and feedback is in register Y_n.

noiseless forward communication available, then

$$C_{0F}(N) = \log \alpha^*(\Gamma).$$

Here, $\Gamma = \Gamma(N)$ is the equivocation (bipartite) graph with an edge xy iff $N(y|x) > 0$, and $\alpha^*(\Gamma)$ is the so-called *fractional packing number* – see Appendix A of the attached paper for its definition, and for a proof of the following result, conjectured by Shannon and proved by Ahlswede [2]:

Proposition 2 *For a bipartite graph Γ on $\mathcal{X} \times \mathcal{Y}$ such that every $x \in \mathcal{X}$ is adjacent to at least one $y \in \mathcal{Y}$,*

$$\begin{aligned}\log \alpha^*(\Gamma) &= C_{\min}(\Gamma)\\ &:= \min\{C(N) : \Gamma(N) \textit{ is a subgraph of } \Gamma\},\end{aligned}$$

where $C(N)$ is the usual Shannon capacity of a noisy classical channel [8].

Note that for (the quantum realisation of) a classical channel, i.e. $\mathcal{N}(\rho) = \sum_{xy} N(y|x)|y\rangle\langle x|\rho|x\rangle\langle y|$, the corresponding subspace is given by

$$K = \operatorname{span}\{|y\rangle\langle x| : xy \text{ is an edge in } \Gamma\},$$

so K should really be understood as the quantum generalisation of the equivocation graph (a *quantum bipartite graph*), much as $S = K^\dagger K$ was advocated in [7] as a quantum generalisation of an undirected graph.

In the following we explore this capacity and an upper bound on it, referring for practically all proofs to the attached full paper.

II. Upper bound on $\mathbf{C_{0EF}}$: a quantum fractional packing number? Recall that for a channel $\mathcal{N} : \mathcal{S}(A) \longrightarrow \mathcal{S}(B)$, the entanglement-assisted classical capacity [4], i.e. the maximum rate of asymptotically error-free communication via many uses of the channel assisted by a suitable pre-shared entangled state, is given by

$$C_E(\mathcal{N}) = \max_\rho I(A:B)_\sigma,$$

where $\sigma_{AB} = (\mathrm{id} \otimes \mathcal{N})\phi_{AA'}$ is the joint input-output state, $\phi_{AA'}$ is a purification of ρ, and $I(A:B) = S(\sigma_A) + S(\sigma_B) - S(\sigma_{AB})$ is the quantum mutual information. Using this, we define for a quantum bipartite graph $K < \mathcal{L}(A \to B)$ such that $\mathbb{1} \in K^\dagger K$ (these are precisely the possible Kraus subspaces of channels):

$$C_{\min E}(K) := \min\{C_E(\mathcal{N}) : \mathcal{K}(\mathcal{N}) < K\}.$$

This definition is of course motivated by Proposition 2, suggesting $2^{C_{\min E}(K)}$ as a possible quantum generalisation of the fractional packing number. For one thing, for the quantum realisation K of a classical equivocation graph Γ, it is easy to see that indeed $C_{\min E}(K) = C_{\min}(\Gamma) = \log \alpha^*(\Gamma)$.

At least, this quantity is related to the feedback-assisted zero-error capacity: the Quantum Reverse Shannon Theorem [3, 5] tell us that $C_E(\mathcal{N})$ is not increased even by allowing feedback, so that $C_{0EF}(K)$ is upper bounded by the entanglement-assisted capacity $C_E(\mathcal{N})$ for any channel $\mathcal{N}$ such that $\mathcal{K}(\mathcal{N}) \subset K$, hence

Lemma 3 *For any quantum bipartite graph $K < \mathcal{L}(A \to B)$ with $\mathbb{1} \in K^\dagger K$, $C_{0EF}(K) \leq C_{\min E}(K)$.*

Some evidence that with our definition we might be on the right track comes from the realisation that $C_{\min E}(K)$ shares many properties with $C_{\min}(\Gamma)$. First, $C_{\min E}(K)$ come from the following results.

Lemma 4 *For any quantum bipartite graph $K < \mathcal{L}(A \to B)$ such that $\mathbb{1} \in K^\dagger K$,*

$$C_{\min E}(K) = \min_{\substack{\mathcal{N} \text{ s.t.}\\ \mathcal{K}(\mathcal{N}) < K}} \max_\rho I(\rho;\mathcal{N}) = \max_\rho \min_{\substack{\mathcal{N} \text{ s.t.}\\ \mathcal{K}(\mathcal{N}) < K}} I(\rho;\mathcal{N}).$$

Lemma 5 *For quantum bipartite graphs $K_1 < \mathcal{L}(A_1 \to B_1)$ and $K_2 < \mathcal{L}(A_2 \to B_2)$ with $\mathbb{1} \in K_i^\dagger K_i$,*

$$C_{\min E}(K_1 \otimes K_2) = C_{\min E}(K_1) + C_{\min E}(K_2).$$

Lemma 6 *Both C_{0EF} and $C_{\min E}$ are monotonic under pre- and post-processing of the channel: for quantum bipartite graphs $K < \mathcal{L}(A \to B)$ and $K_A < \mathcal{L}(A_0 \to A)$, $K_B < \mathcal{L}(B \to B_0)$, note that the matrix-multiplied space $K_B K K_A < \mathcal{L}(A_0 \to B_0)$ is a quantum bipartite graph. Then,*

$$C_{0EF}(K) \geq C_{0EF}(K_B K K_A),$$
$$C_{\min E}(K) \geq C_{\min E}(K_B K K_A).$$

III. $\mathbf{C}_{\min \mathbf{E}}(\mathbf{K})$ equals the feedback-assisted conclusive capacity. To make the statement in this section's title a theorem, we need to give a definition of this capacity. The idea of it is that the sender and receiver may use arbitrary entanglement and feedback strategies to transmit classical messages through $n \to \infty$ many uses of the channel, and at the end of the protocol they need to either succeed in transmitting the message, or conclude that they failed, which must not happen with probability larger than $\epsilon \to 0$. Here, "n uses of the channel" refers to an *arbitrary* channel (cptp map) $\mathcal{N}^{(n)}$ such that $\mathcal{K}(\mathcal{N}^{(n)}) < K^{\otimes n}$. This channel may be chosen adversarially, and a code has to use the resources available in such a way that the maximum decoding error probability over all $\mathcal{N}^{(n)}$ is bounded by ϵ. Then, with the maximum number of messages in a code denoted $M(n, \epsilon)$, we may define

$$C_{*EF}(K) := \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log M(n, \epsilon).$$

Theorem 7 *For any quantum bipartite graph $K < \mathcal{L}(A \to B)$ with $\mathbb{1} \in K^\dagger K$, $C_{*EF}(K) = C_{\min E}(K)$.*

With this result as our best evidence that indeed $C_{0EF}(K) = C_{\min E}(K)$ in general (assuming availability of a finite amount of zero-error communication, as in the classical case), we close this section.

IV. Examples: Weyl-diagonal and cq-channels.

A. Denoting by X and Z the discrete translation and phase shift, consider the channel $\mathcal{N}_0(\rho) = \sum_{a,b=0}^{d-1} p_{ab} X^a Z^b \rho Z^{-b} X^{-a}$, with probabilities $p_{ab} \geq 0$ summing to 1. Clearly,

$$K = \mathcal{K}(\mathcal{N}_0) = \operatorname{span}\{W_{ab} := X^a Z^b \,:\, p_{ab} > 0\}.$$

One can prove that

$$C_{\min E}(K) = 2 \log d - \log k = C_{0EF}(K).$$

B. For a given orthonormal basis $\{|i\rangle\}$ of the input space, and pure states $|\psi_i\rangle$ in the output space, consider the cq-channel $\mathcal{N}_0(\rho) = \sum_i |\psi_i\rangle\langle i|\rho|i\rangle\langle\psi_i|$, which has Kraus subspace

$$K = \mathcal{K}(\mathcal{N}_0) = \operatorname{span}\{|\psi_i\rangle\langle i|\},$$

and one can show that

$$C_{\min E}(K) = \max_{(p_i)\text{ p.d.}} S\left(\sum_i p_i |\psi_i\rangle\langle\psi_i|\right).$$

Note that this can vary arbitrarily between 0 and $\log |A|$ even when $K^\dagger K = \mathcal{L}(A)$ (i.e. all $|\psi_i\rangle$ are pairwise non-orthogonal).

Proposition 8 *$C_{0EF}(K)$ – when assisted by a finite amount of noiseless forward communication – is always positive if $C_{\min E}(K)$ is; i.e. if the $|\psi_i\rangle$ are not all collinear. Furthermore, $C_{0EF}(K)$ depends nontrivially on the geometry of the vector arrangement, even if they are all pairwise non-orthogonal: when they are close to parallel, $C_{0EF}(K)$ is arbitrarily close to 0, when they are sufficiently close to being mutually orthogonal, $C_{0EF}(K)$ is arbitrarily close to $\log |A|$.*

We have also investigated a class of cq-channels with mixed output states and showed that C_{0EF} for these is positive, but refer the reader to the attached full paper for details.

V. Conclusion. We have introduced the problem of determining the zero-error capacity of a quantum channel assisted by noiseless feedback, for which we introduce an interesting upper bound, achievable under certain relaxations of the problem.

We consider our work here mainly as opening up the problem, and close highlighting the open questions and possible next steps:

1. Is $C_{\min E}(K)$ always an achievable ffedback-assisted zero-error rate? Is even $C_{0EF}(K) > 0$ whenever $C_{\min E}(K) > 0$? The cq-channels considered above seem to offer a good testing ground for ideas.

2. Is there a manifestly linear or semidefinite programming (or even just convex optimisation) characterisation of $2^{C_{\min E}(K)}$? To make progress, we need at least to understand some properties of an optimal $\mathcal{N}$ for given K, and potentially also an optimal input state.

3. What is the zero-error simulation cost of a fixed channel $\mathcal{N}$? Minimum cost over all channels with $\mathcal{K}(\mathcal{N}) < K$? (Cf. results along these lines for classical channels in [6].)

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