(Non-)Contextuality of Physical Theories as an Axiom

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Plan

1. **Introduction:**

non-contextuality

2. **Results:**

a general framework to study non-contextuality;
non-contextuality and non-locality

3. **Open problems:**

theoretical;
applied;
a complexity perspective.

1. Introduction:

non-contextuality for

1. classical theories
2. non-signaling theories
3. quantum theory

“measuring”

means

“to ask questions to a system”

“measuring”

means

“to ask questions to a system”

what is your velocity?

“measuring”

means

“to ask questions to a system”

what is your velocity?

what is your angular momentum?

“measuring”

means

“to ask questions to a system”

what is your velocity?

what is your angular momentum?

what is the colour of your eyes?

“measuring”

means

“to ask questions to a system”

what is your velocity?

what is your angular momentum?

what is the colour of your eyes?

is your entropy 5 bits?

“measuring”

means

“to ask questions to a system”

what is your velocity?

what is your angular momentum?

what is the colour of your eyes?

is your entropy 5 bits?

Question 1

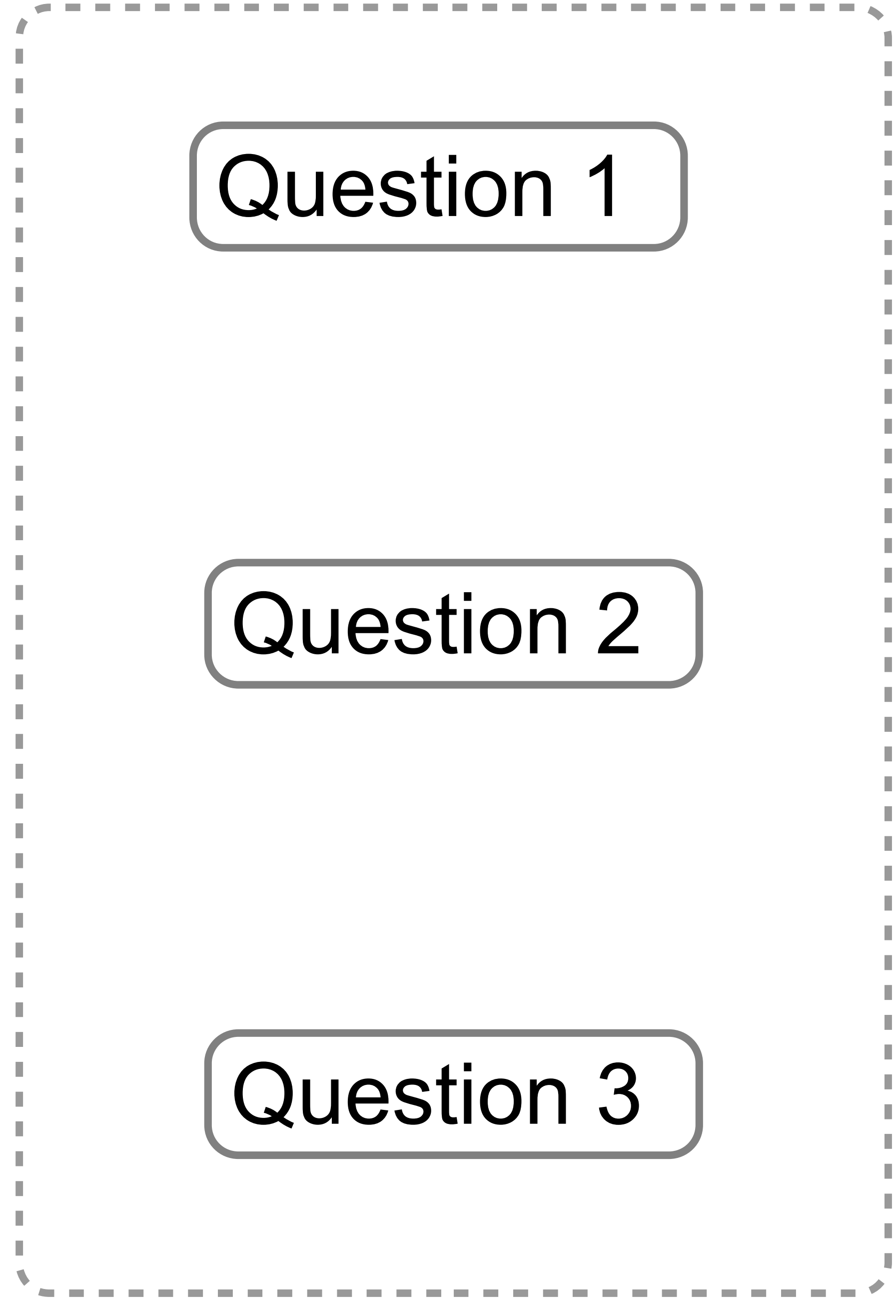
Question 5

Question 2

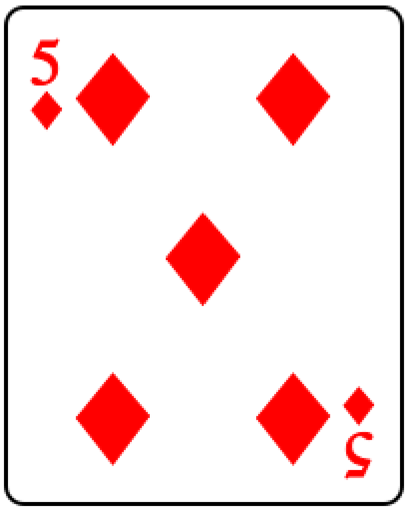
Question 4

Question 3

<u>context</u>:
a set of
mutually compatible
questions



compatibility:



Question 1:
Is the colour red?
Answer:
Yes

Question 2:
What is the suit?
Answer:
Diamonds

non-contextuality:

answers

do not depend

on contexts

Context 1

Question 1:
Is X true?
Answer:
Yes

Question 2:
Is Y true?
Answer:
Yes

non-contextuality:

answers
do not depend
on contexts

Context 2

Question 2:
Is Y true?
Answer:
Yes

Question 3:
Is Z true?
Answer:
No

<u>non-contextuality</u>:
answers
do not depend
on contexts

Context 2

Context 1

Question 1:
Is X true?
Answer:
Yes

Question 2:
Is Y true?
Answer:
Yes

<u>non-contextuality</u>:
answers
do not depend
on contexts

Question 3:
Is Z true?
Answer:
No

Context 1

Context 2

Question 1:
Is X true?
Answer:
Yes

Question 2:
Is Y true?
Answer:
<u>Yes</u>

Question 3:
Is Z true?
Answer:
No

<u>non-contextuality</u>:
answers
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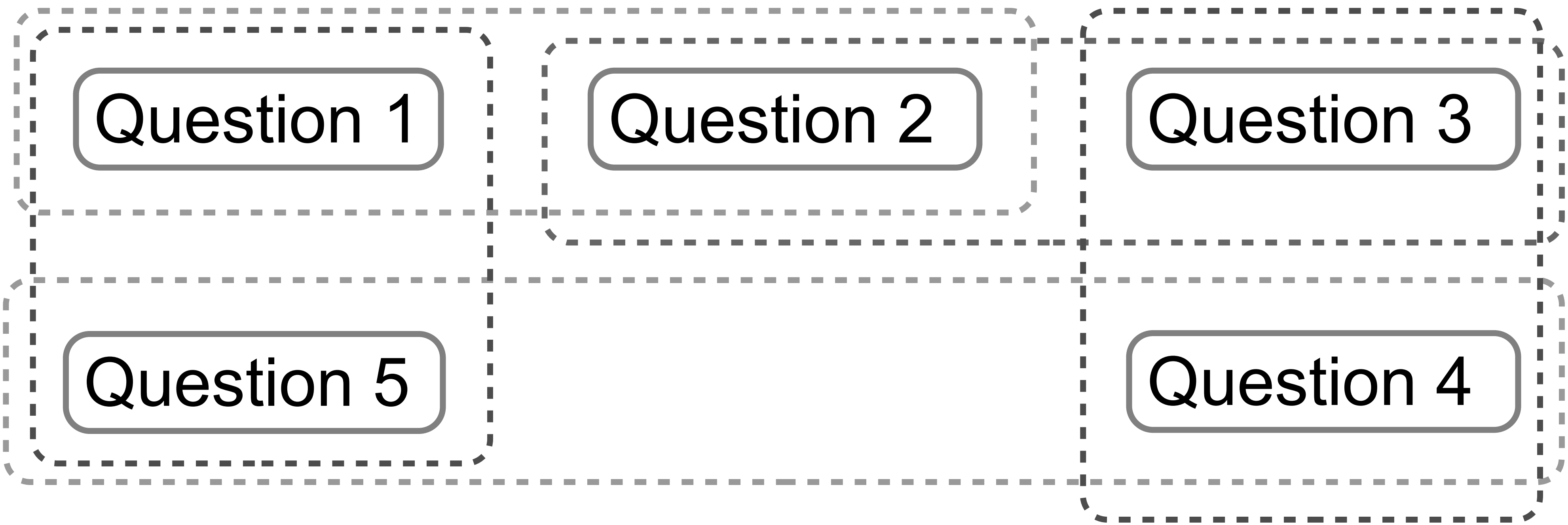
compatibility structure



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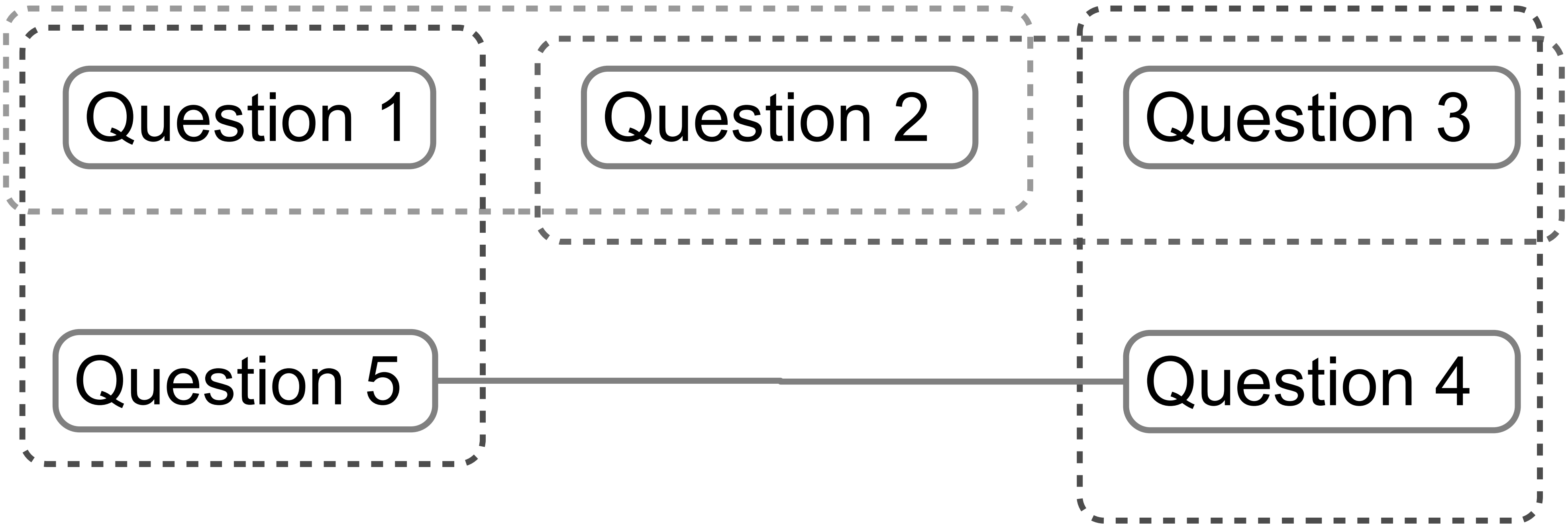

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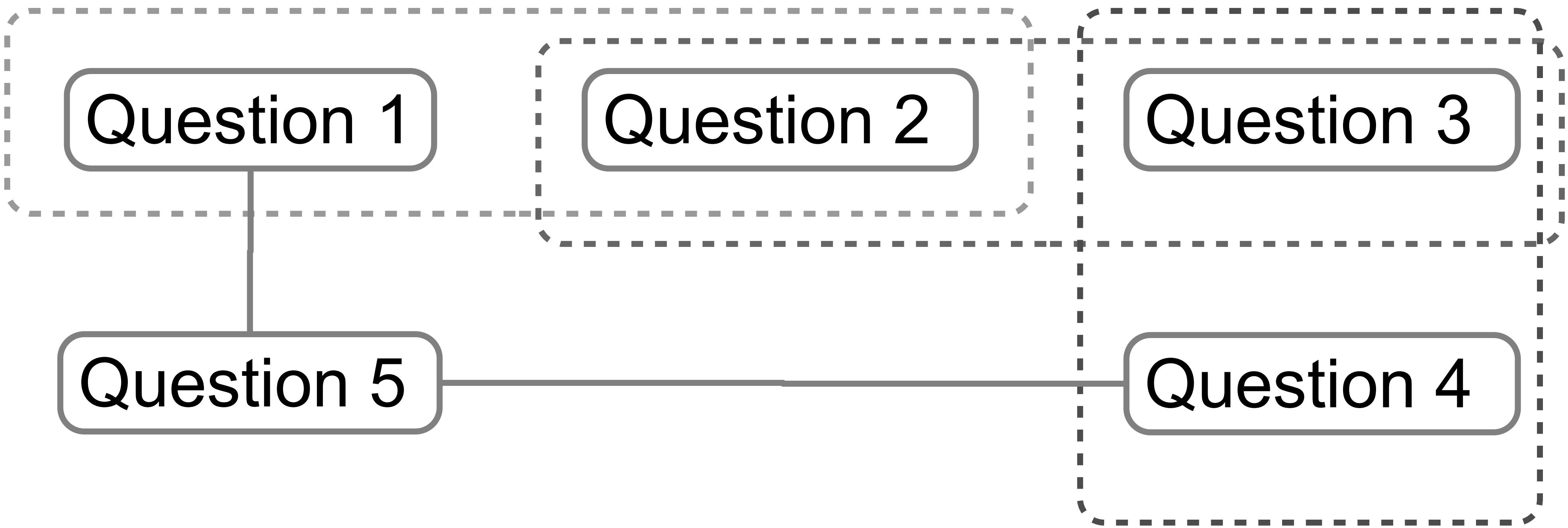

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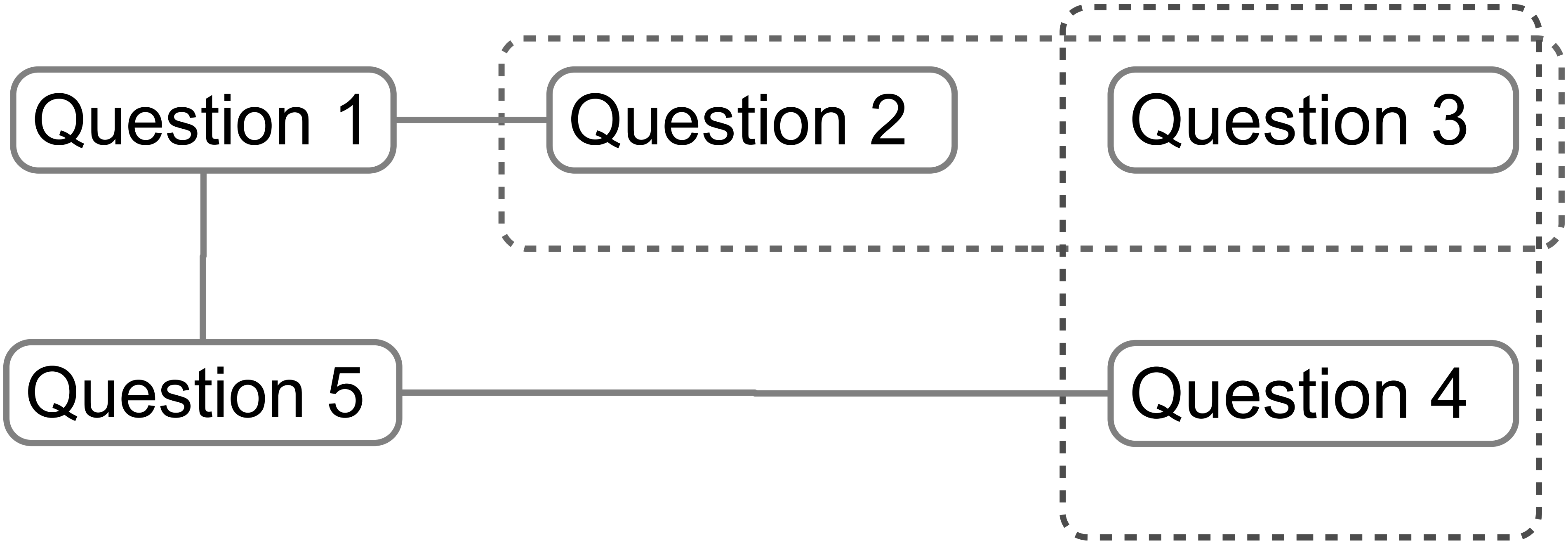
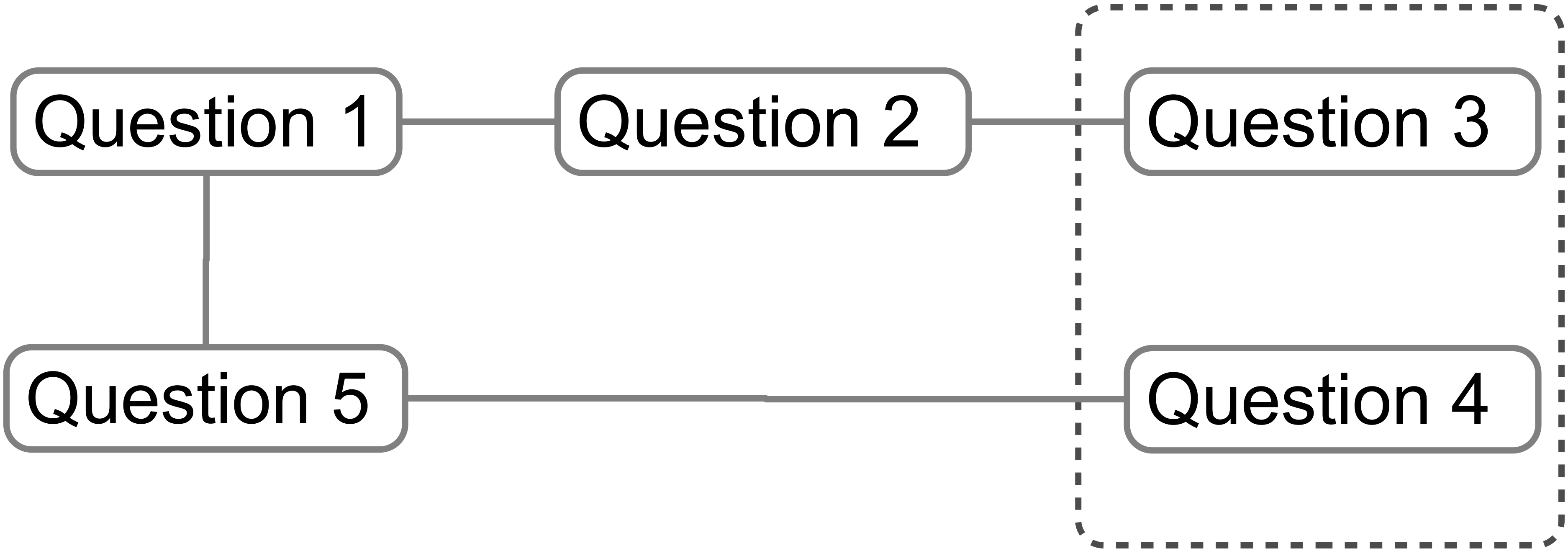
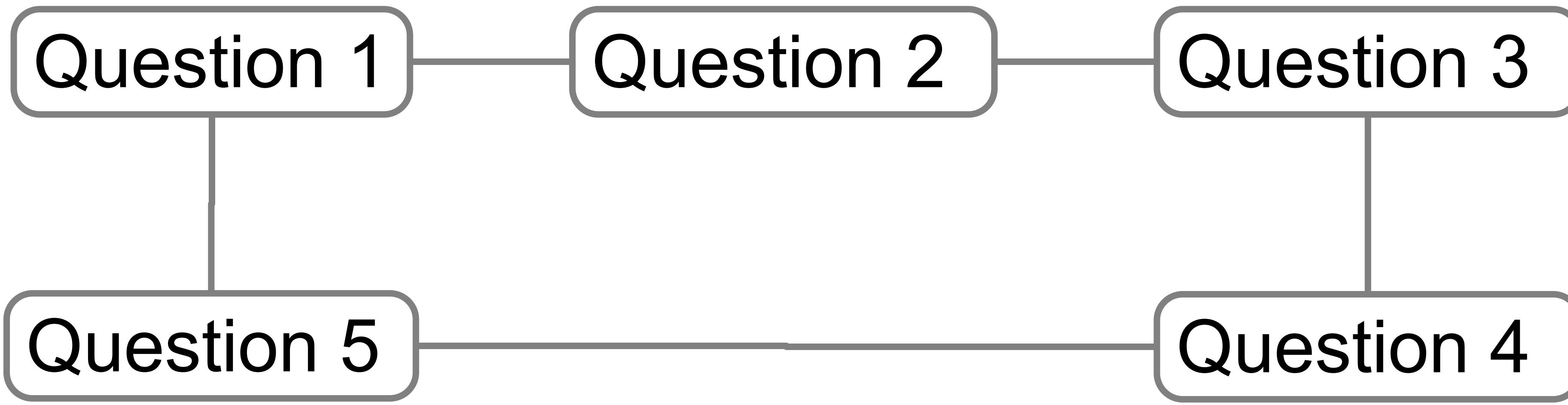


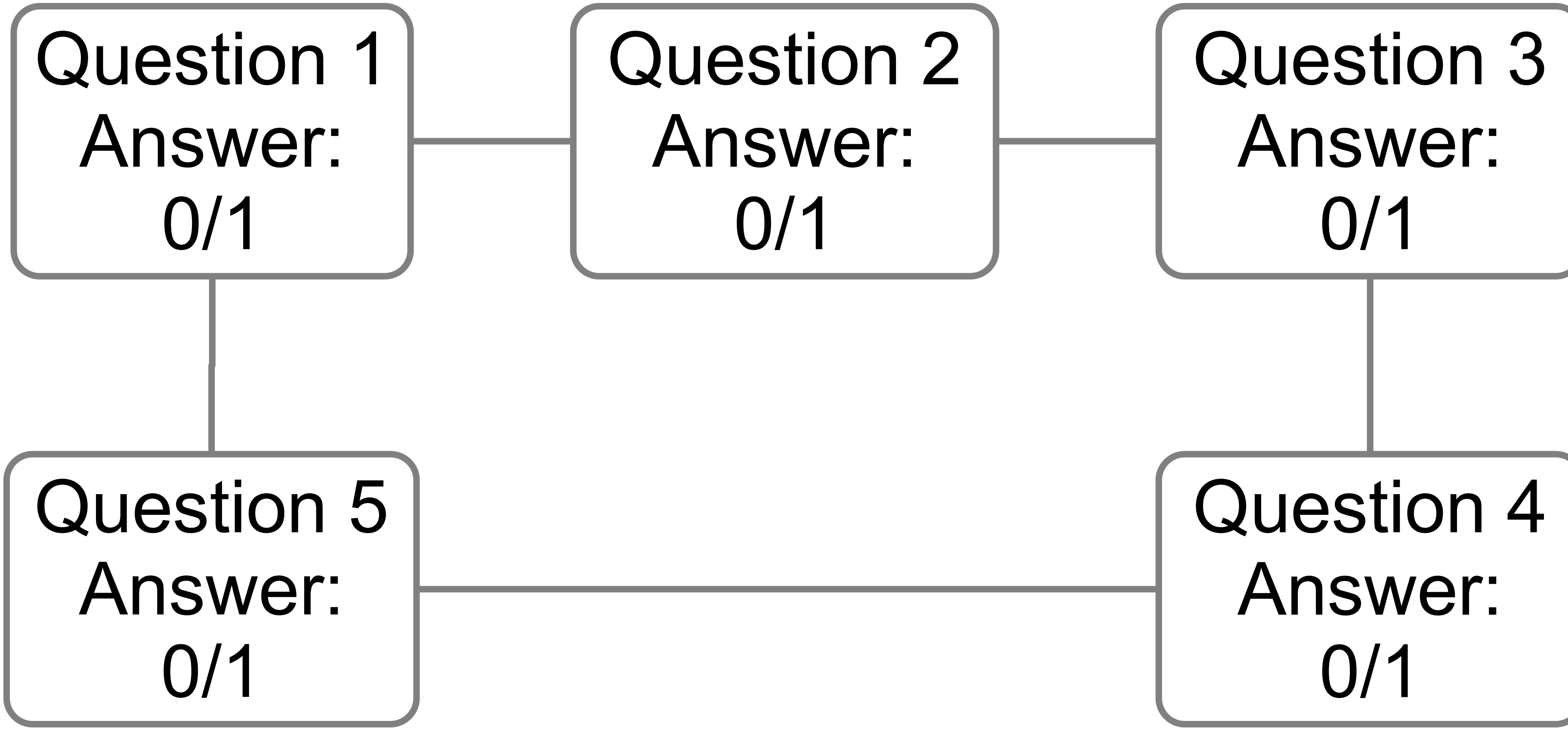
compatibility structure



compatibility structure



exclusiveness:
answers to
adjacent questions
are **not both 1**



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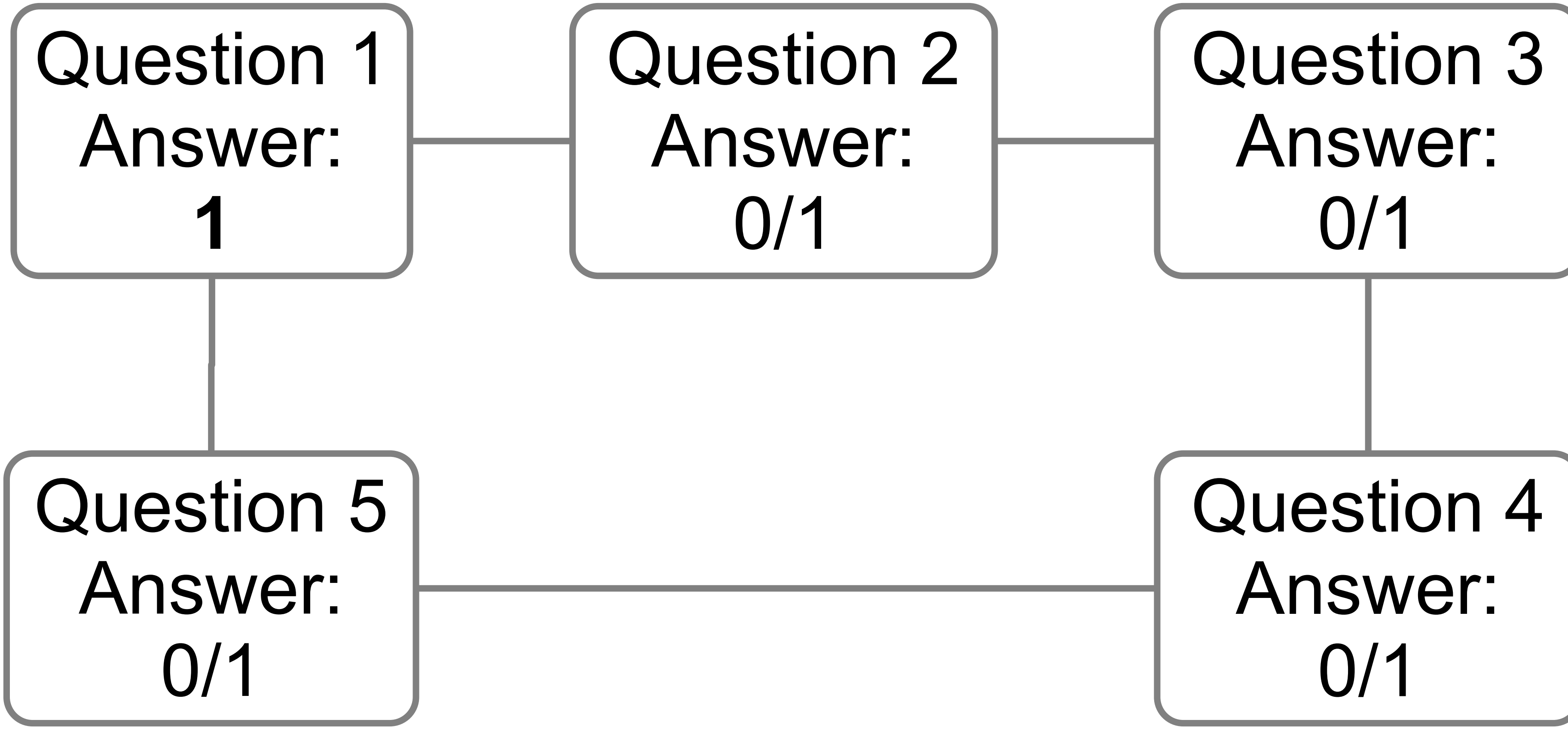

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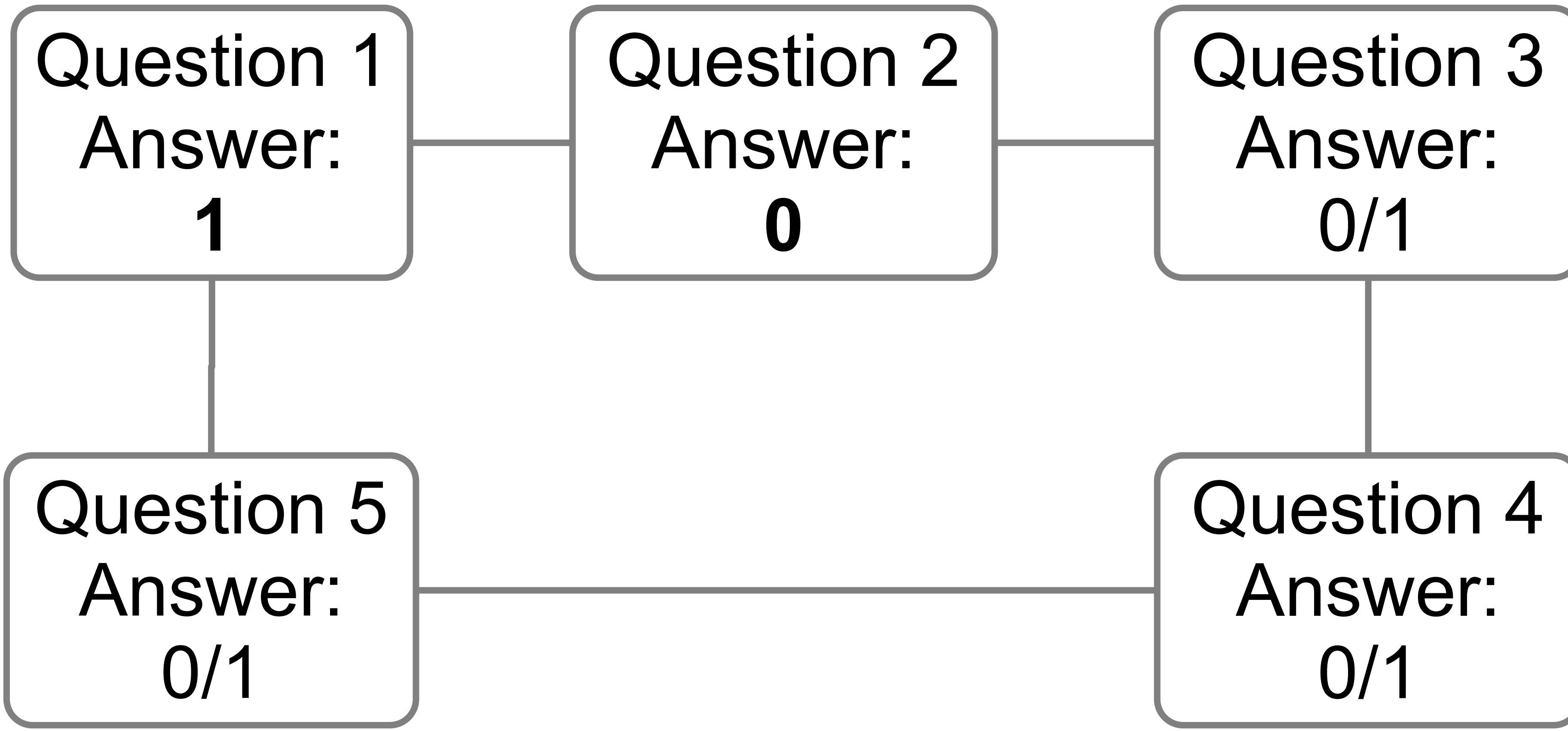

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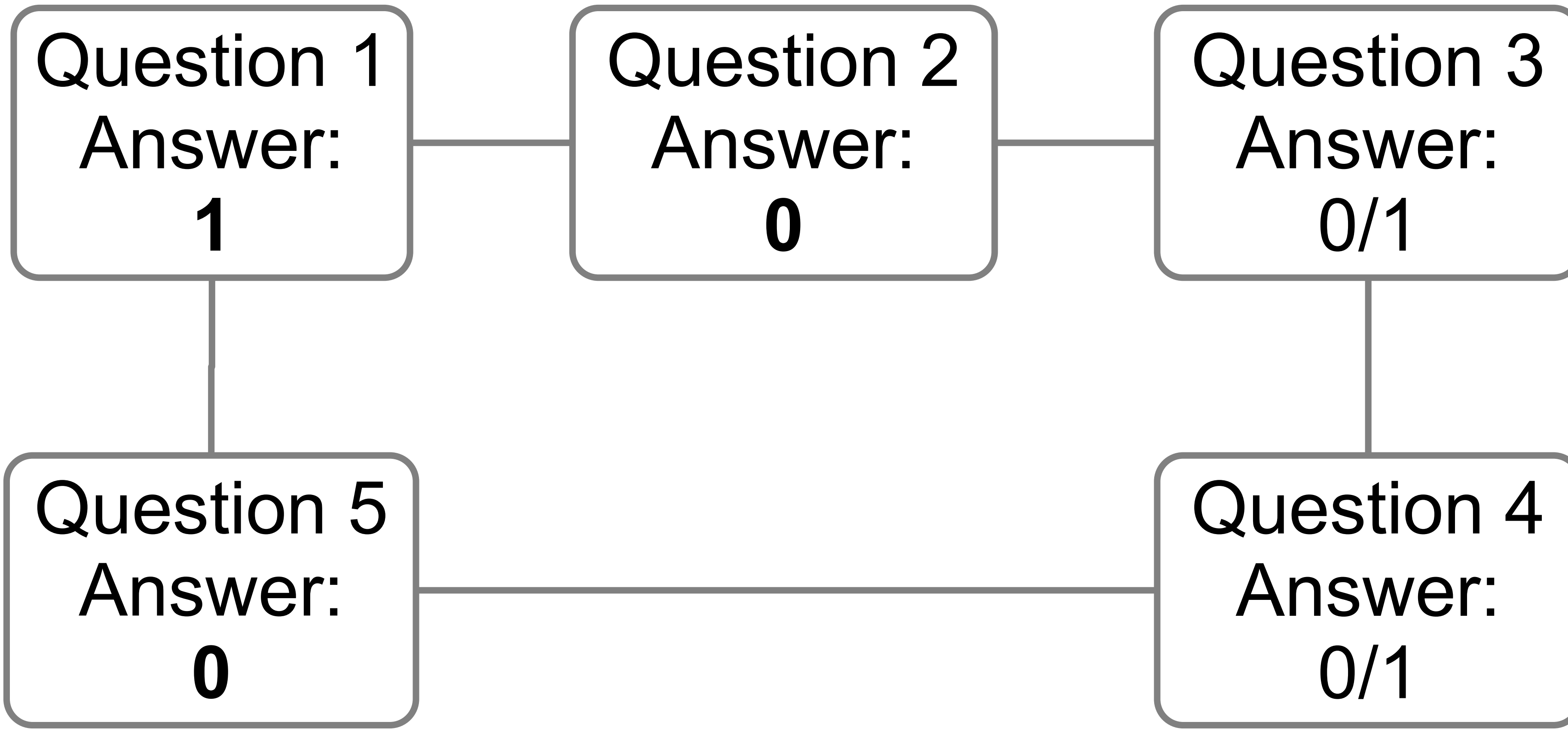

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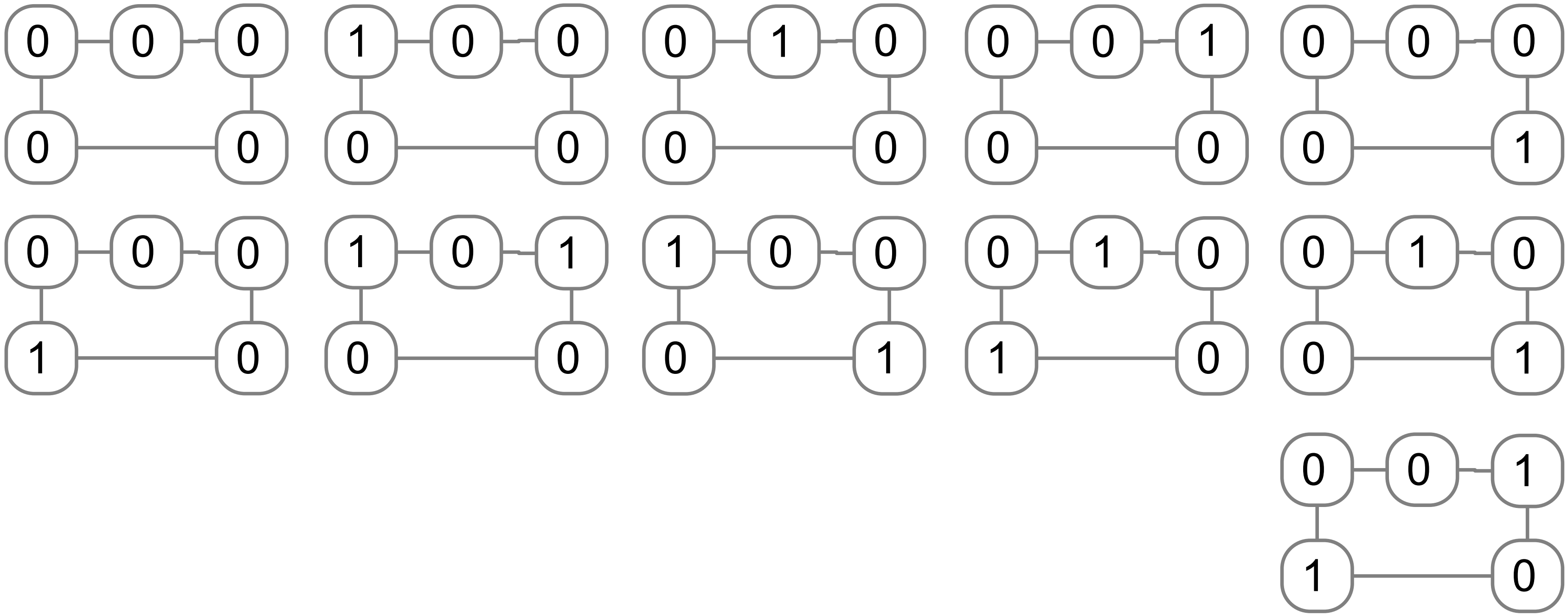


the answers 1 have expectation ≤ 2
if non-contextual and exclusive

classical theories
give expectation ≤ 2

non-contextuality:
probabilities of answers
do not depend
on contexts

Context 1

Question 1:
Is X true?

Pr[answer Yes] = *a*

Question 2:
Is Y true?

Pr[answer Yes] = *b*

non-contextuality:
probabilities of answers
do not depend
on contexts

non-contextuality:
probabilities of answers **do not** depend on contexts

Context 2

Question 2:
Is Y true?

Pr[answer Yes] = *b*

Question 3:
Is Z true?

Pr[answer Yes] = *c*

Context 1

Context 2

Question 1:
Is X true?

Pr[answer Yes] = *a*

Question 2:
Is Y true?

Pr[answer Yes] = *b*

Question 3:
Is Z true?

Pr[answer Yes] = *c*

non-contextuality:
probabilities of answers
do not depend
on contexts

Context 2

Context 1

Question 1:
Is X true?

Pr[answer Yes] = *a*

Question 2:
Is Y true?

Pr[answer Yes] = *b*

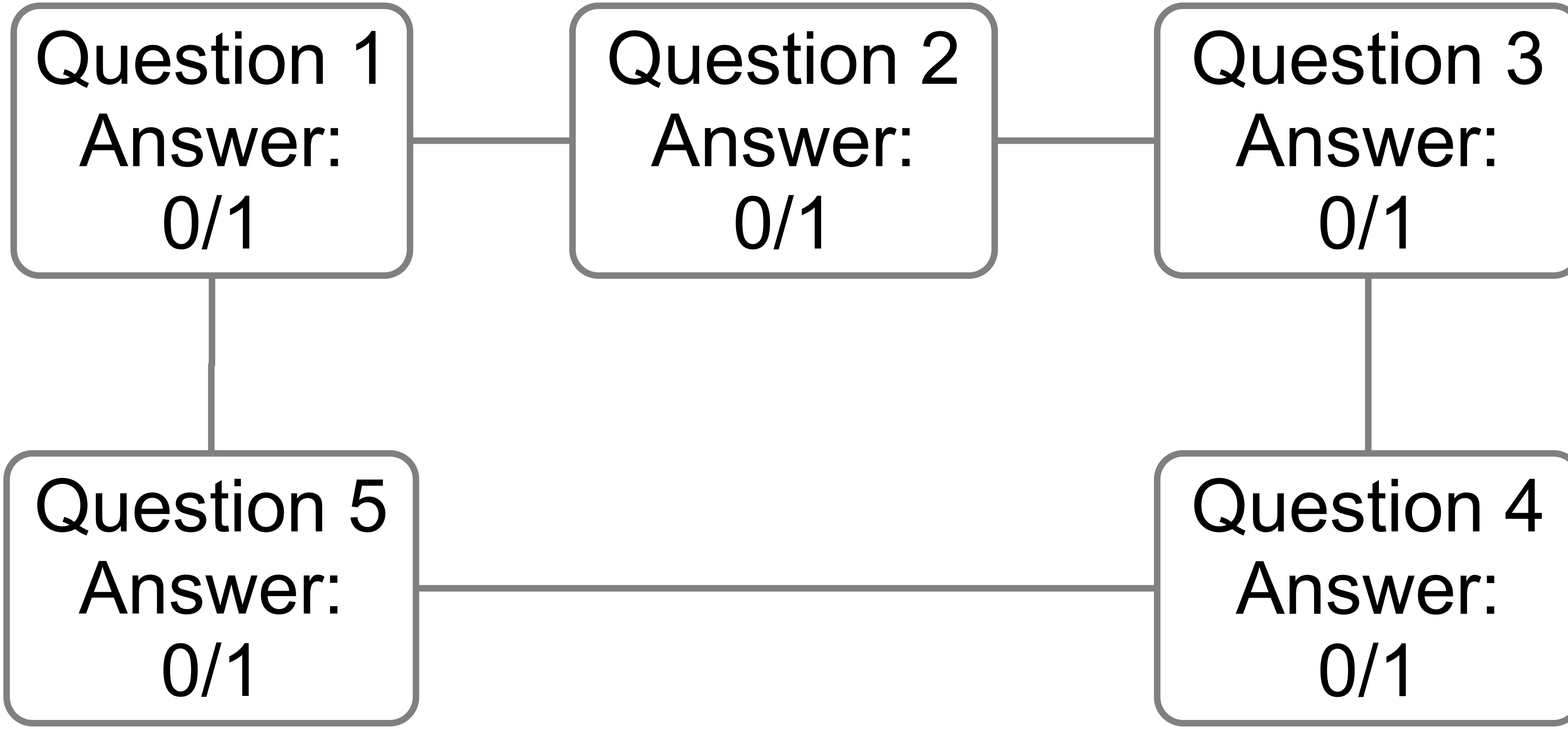
Question 3:
Is Z true?

Pr[answer Yes] = *c*

non-contextuality:
probabilities of answers
do not depend
on contexts

exclusiveness:

answers to adjacent questions are **not both 1**



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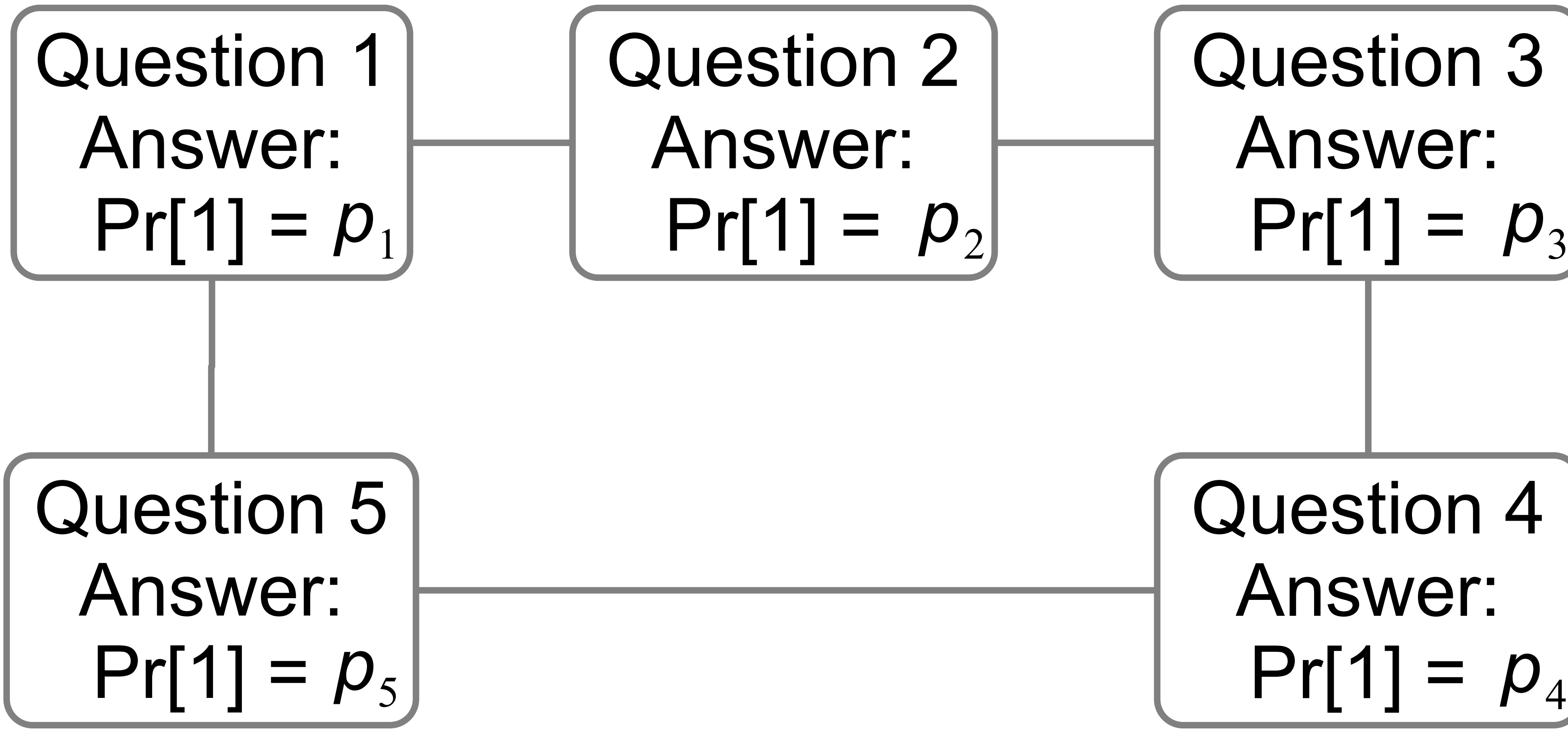

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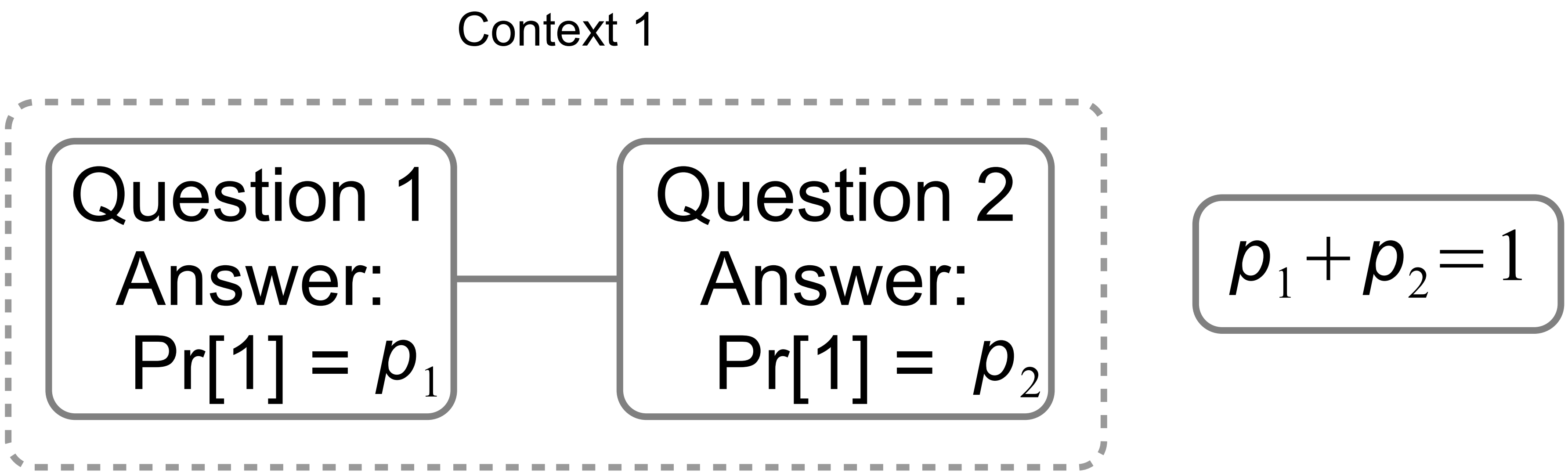

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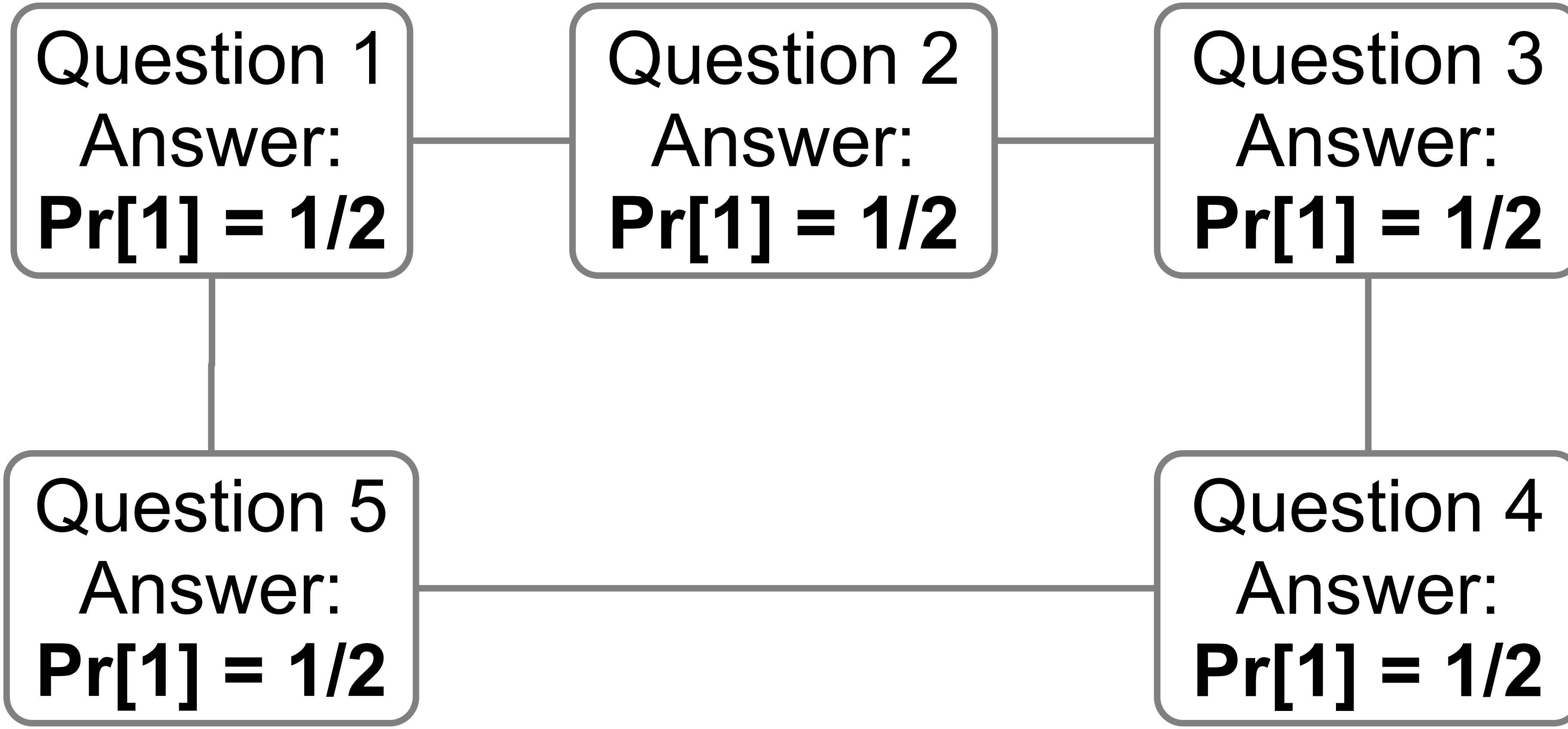


exclusiveness:
answers to
adjacent questions
are **not both 1**

non-signaling theories*
give expectation ≤ 2.5

*also called “general probabilistic theories”

axiomatically:

classical theories
give expectation ≤ 2

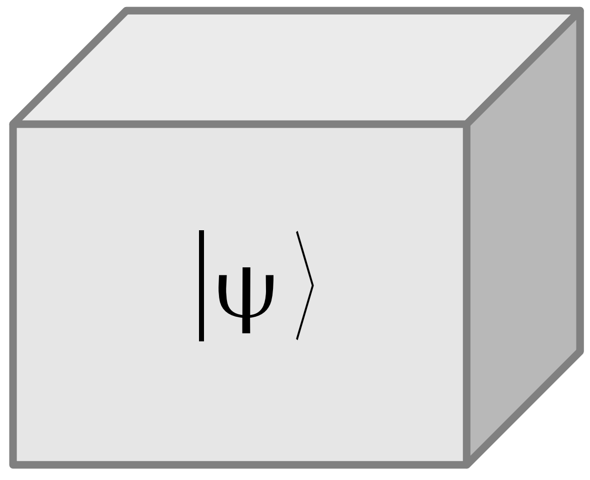
non-signaling theories
give expectation ≤ 2.5

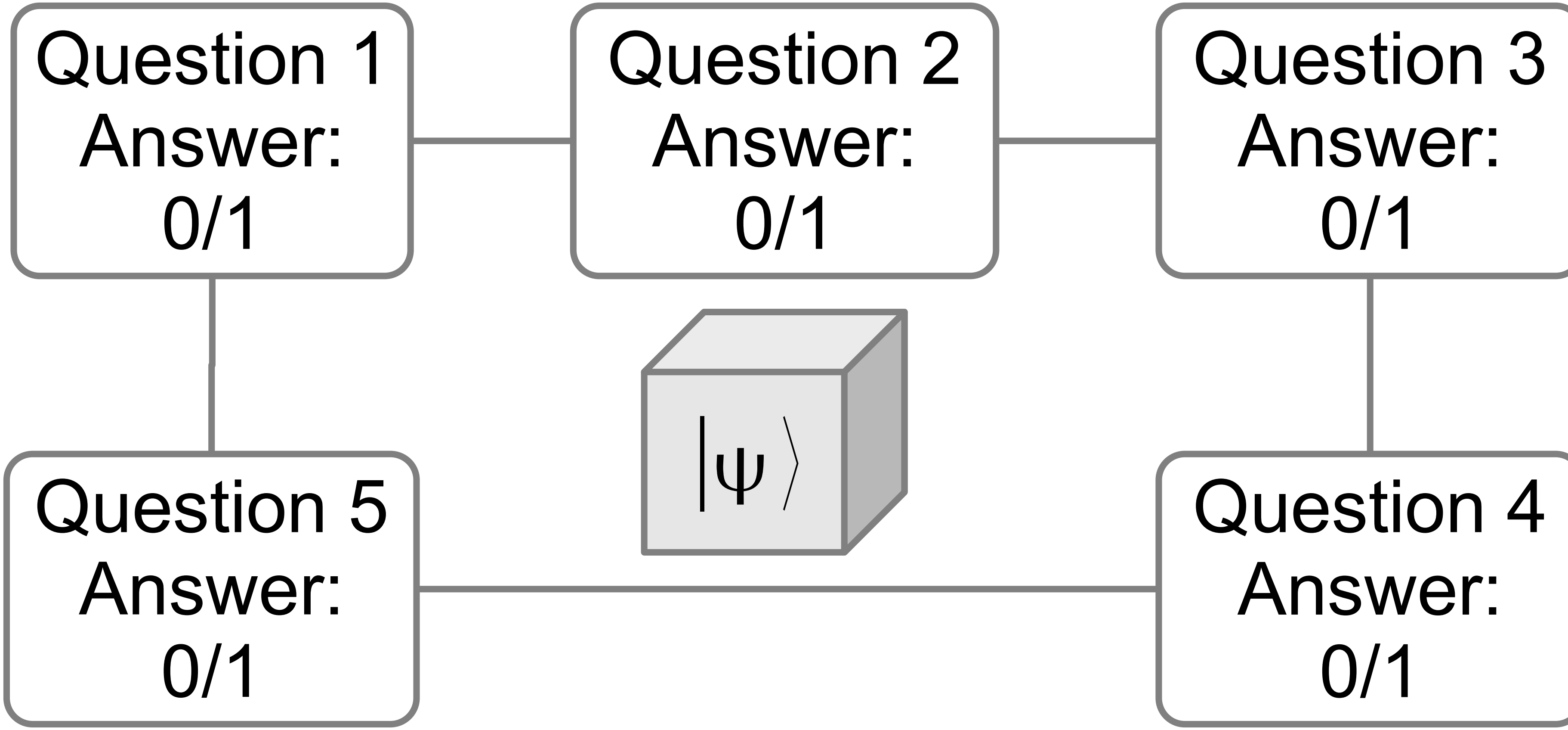
axiomatically:

classical theories
give expectation ≤ 2

quantum theory
give expectation ≤ ?

non-signaling theories
give expectation ≤ 2.5


$|\psi\rangle$



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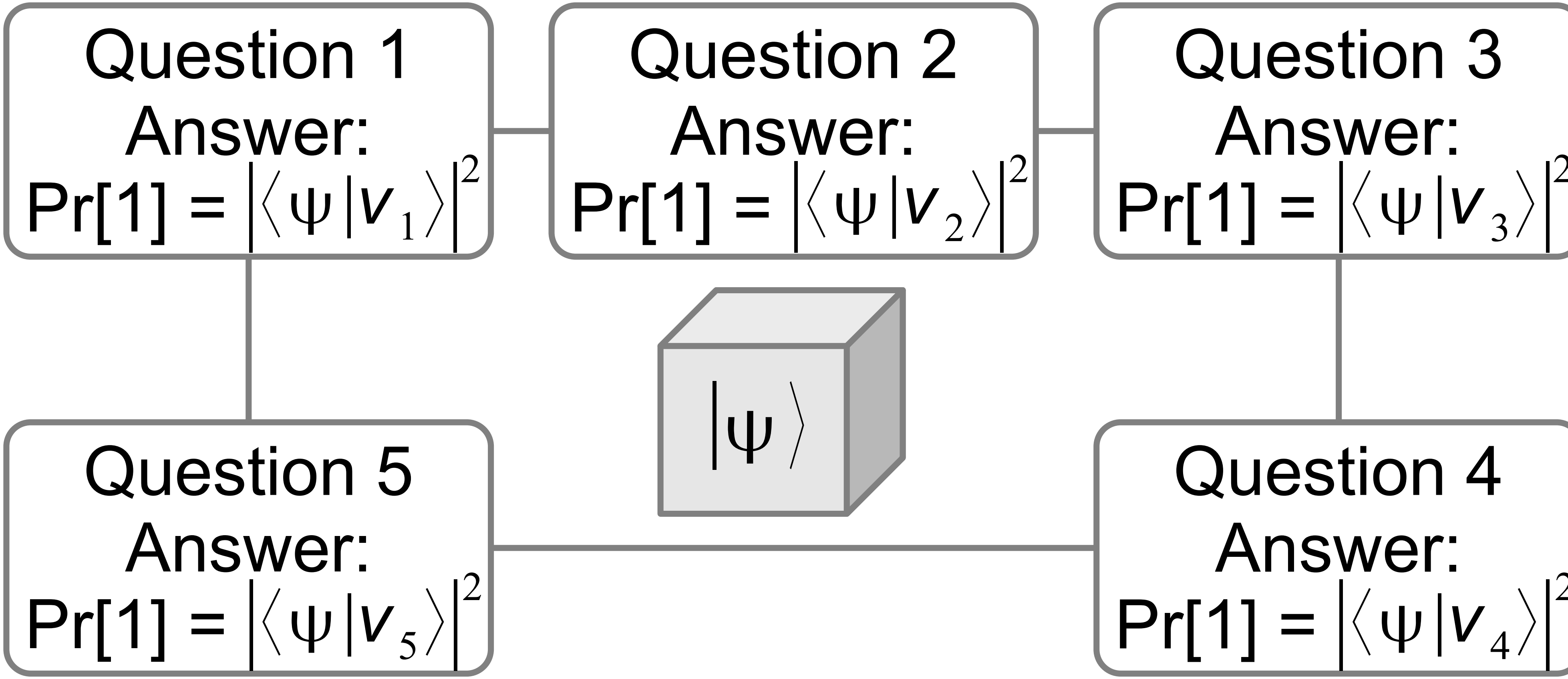


extra axiom:
Born rule

expectation:

$$\sum_{i=1}^{5} |\langle \psi | v_i \rangle|^2$$

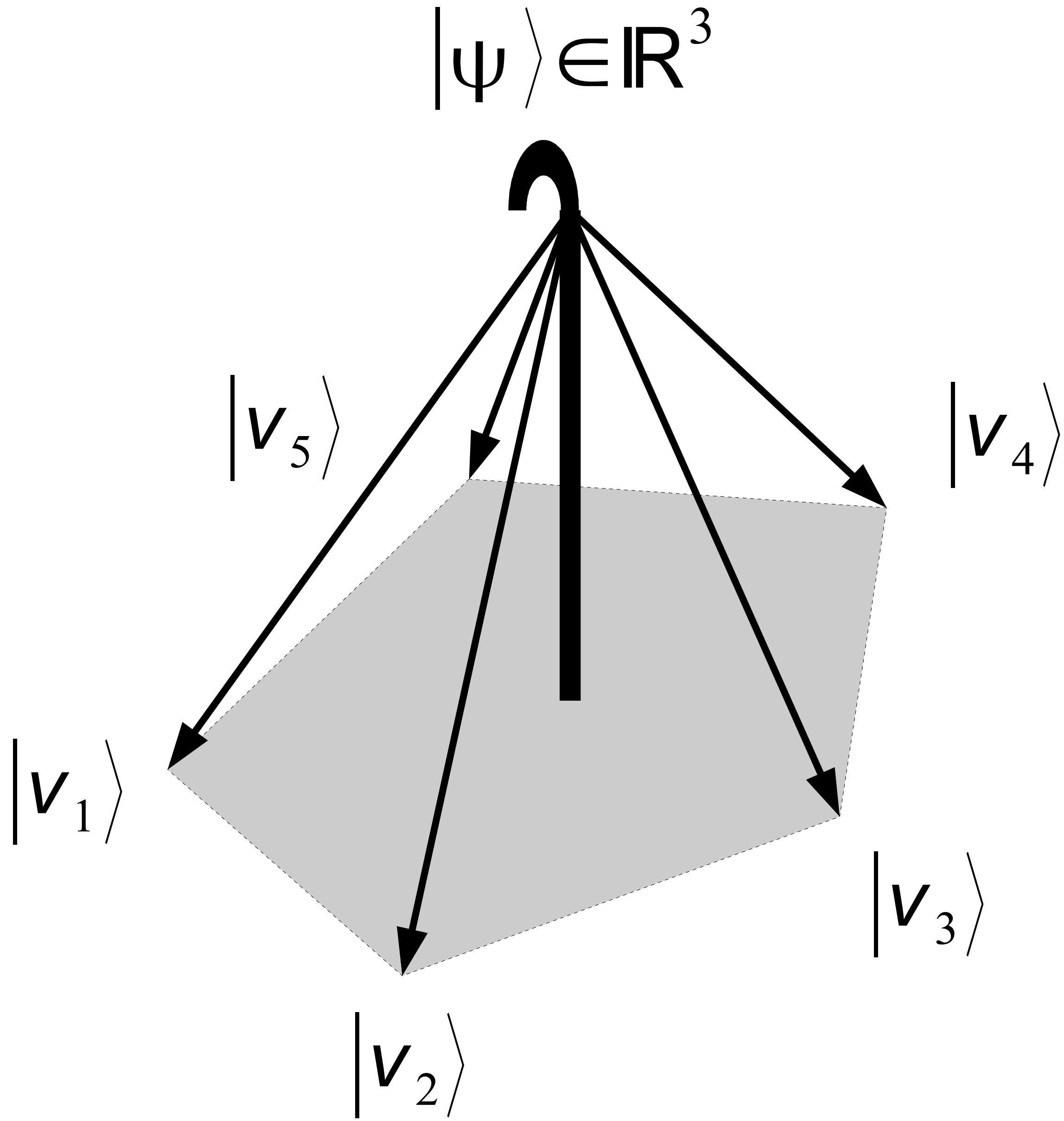
Context i

Question i
Answer:
$$\Pr[1] = \left|\langle \psi | v_i \rangle\right|^2$$

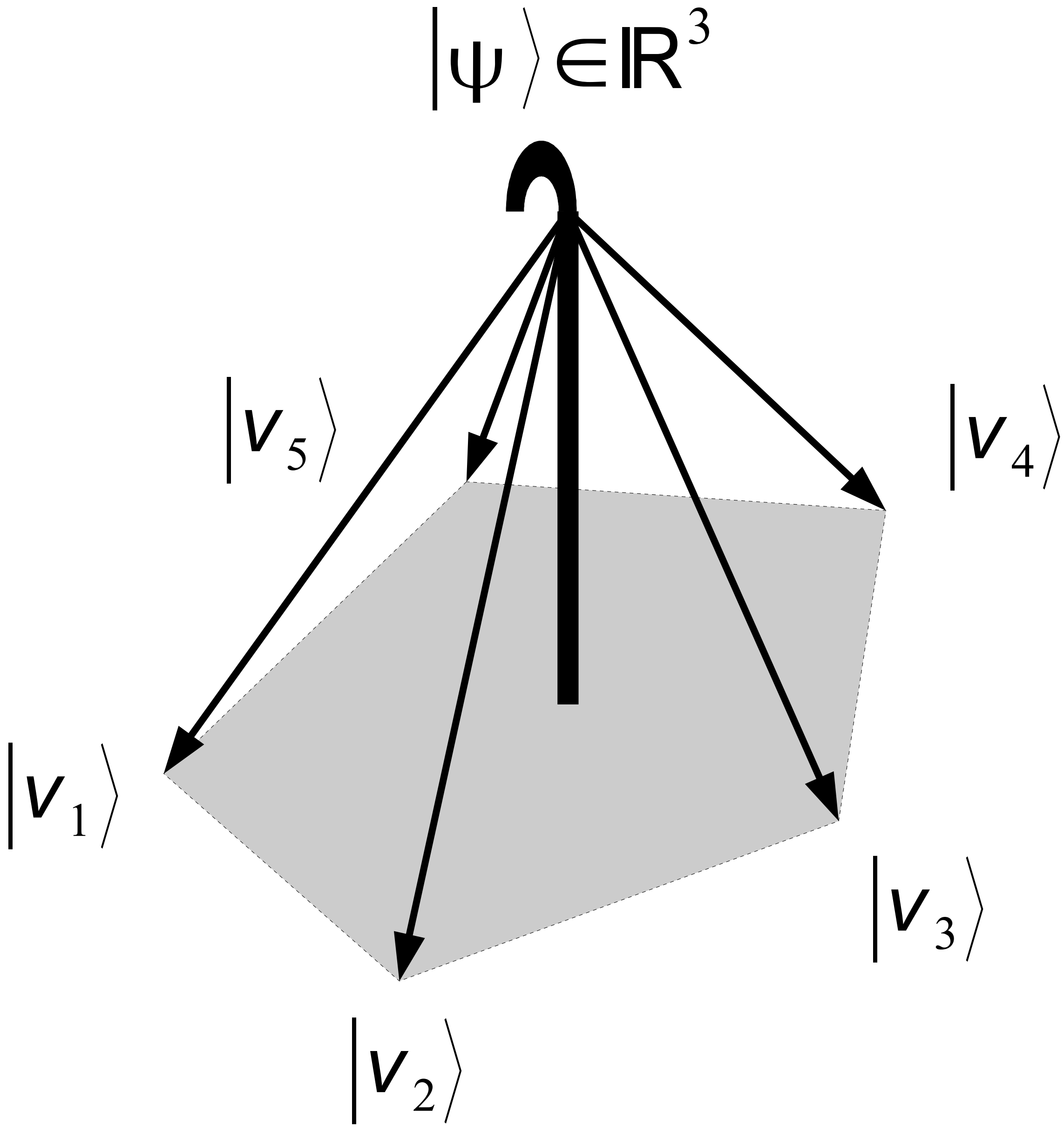
Question i + mod(5)
Answer:
$$\Pr[1] = \left|\langle \psi | v_{i+1\,\mathrm{mod}(5)} \rangle\right|^2$$

$$\langle v_i | v_{i+1\,\mathrm{mod}(5)} \rangle = 0$$

<u>compatibility</u>



$$\left|\langle \psi | v_i \rangle\right|^2 = \frac{1}{\sqrt{5}}$$



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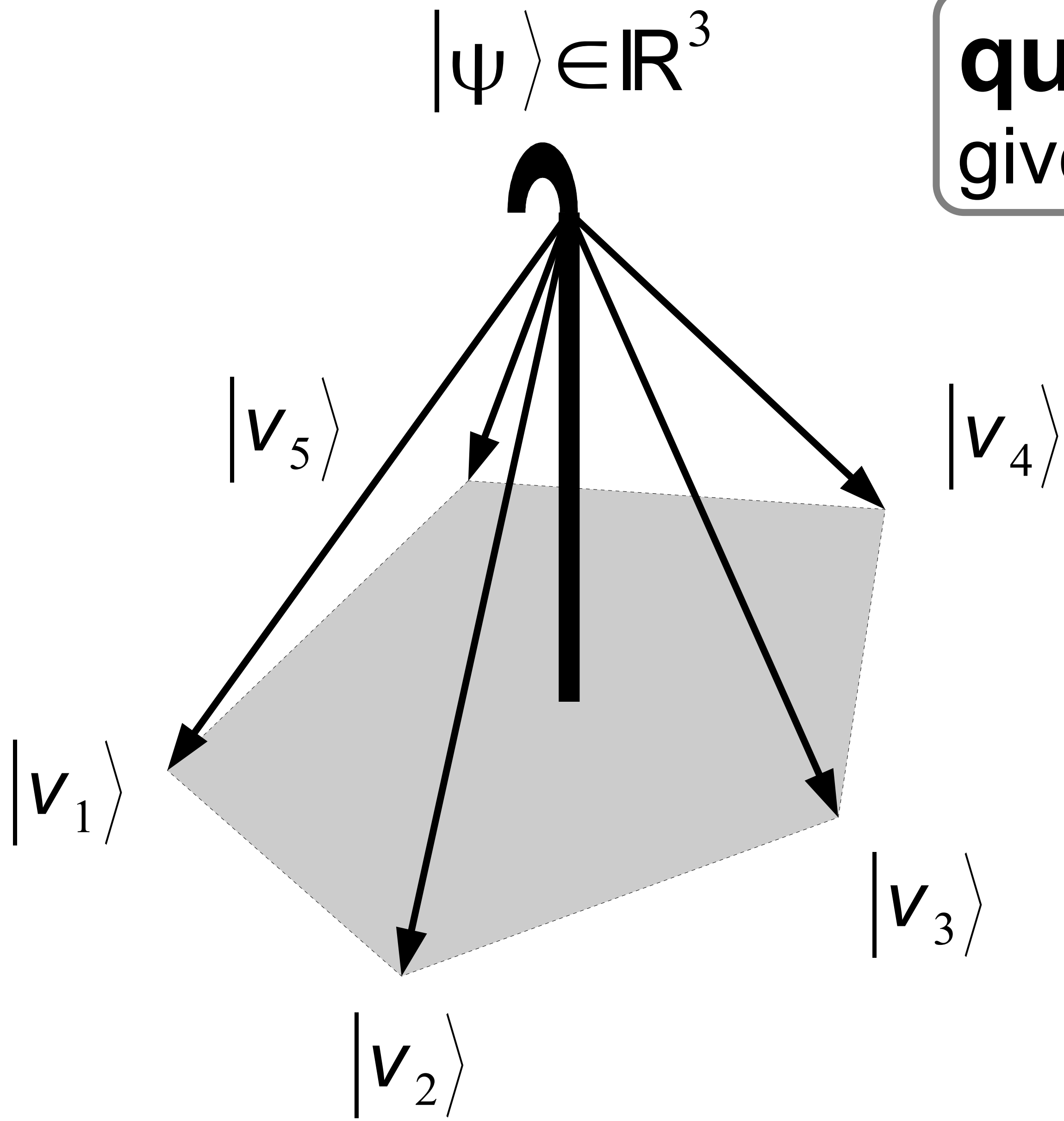
$$\sum_{i=1}^{5} \left|\langle \psi | v_i \rangle\right|^2 = \sqrt{5}$$



quantum theory
gives expectation ≤ $\sqrt{5}$

$$\left|\langle \psi | v_i \rangle\right|^2 = \frac{1}{\sqrt{5}}$$

$$\sum_{i=1}^{5} \left|\langle \psi | v_i \rangle\right|^2 = \sqrt{5}$$

classical theories*
give expectation ≤ 2

quantum theory**
gives expectation ≤ $\sqrt{5} \approx 2.23$

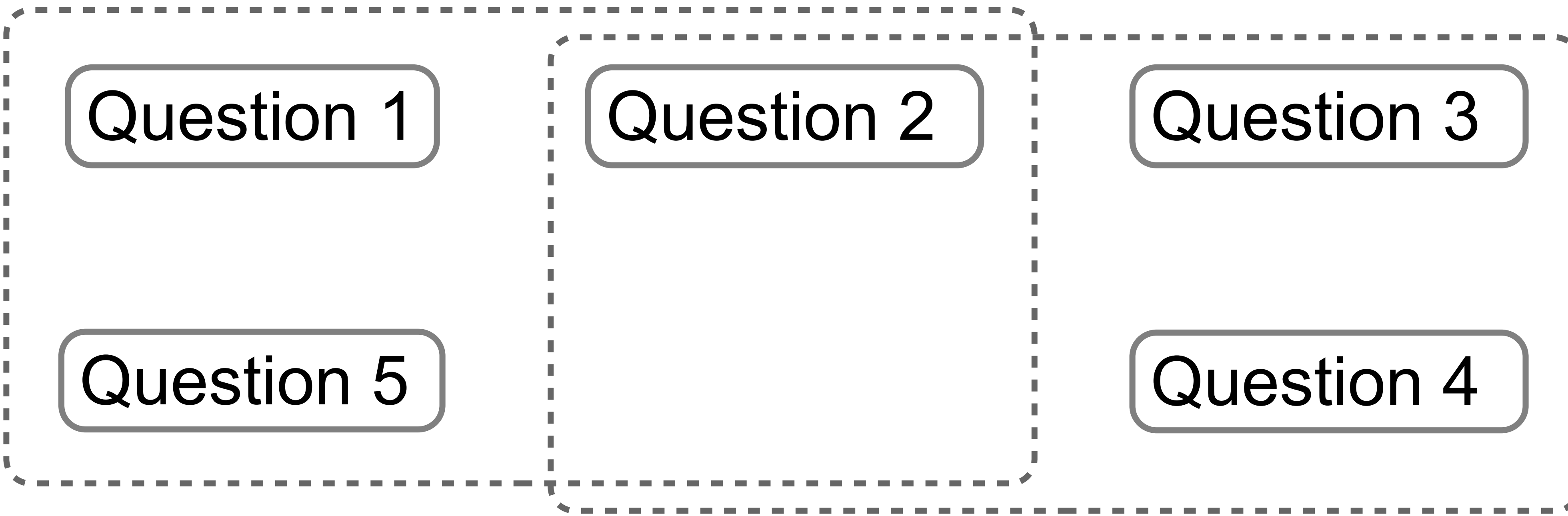
non-signaling theories*
give expectation ≤ 2.5

*Wright (1978); **Klyachko *et al.* (2008)

2. Results:

a general framework
to study non-contextuality:
1. general compatibility structures
2. perfectness
3. non-locality

every graph/hypergraph is a compatibility structure



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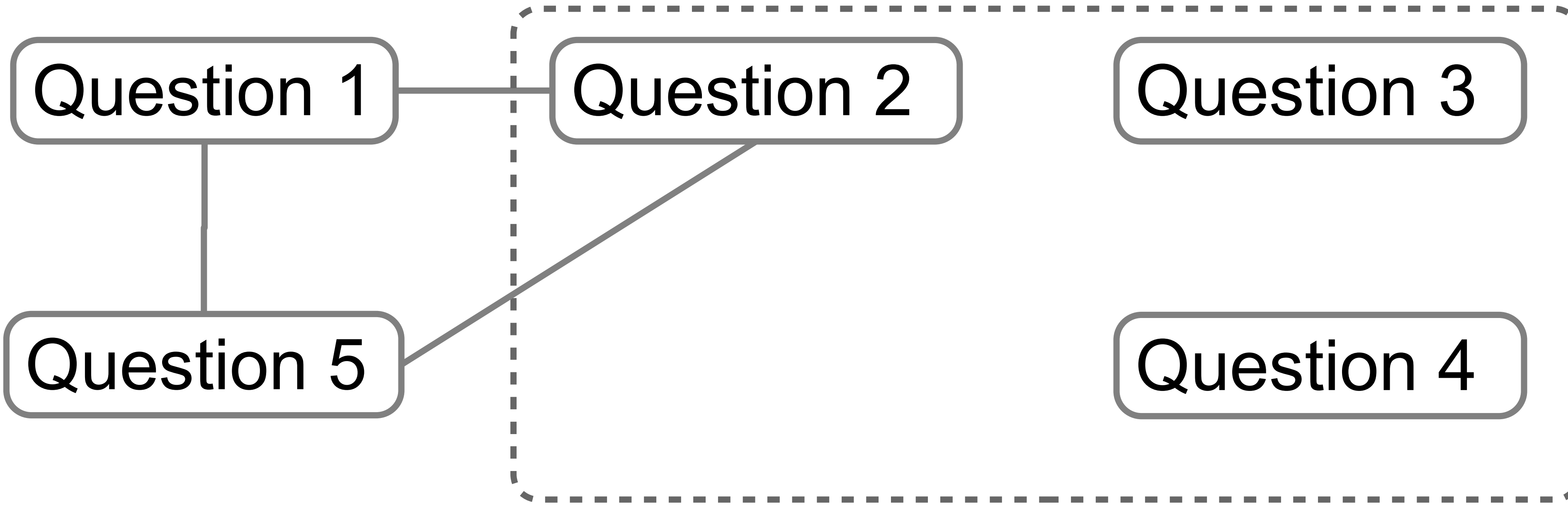

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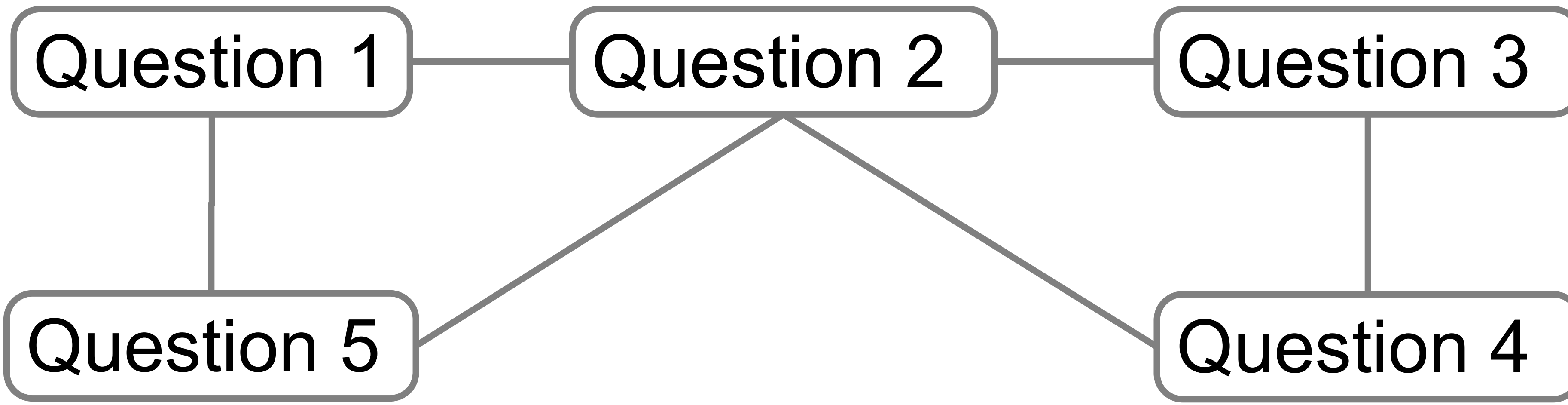


Classification theorem

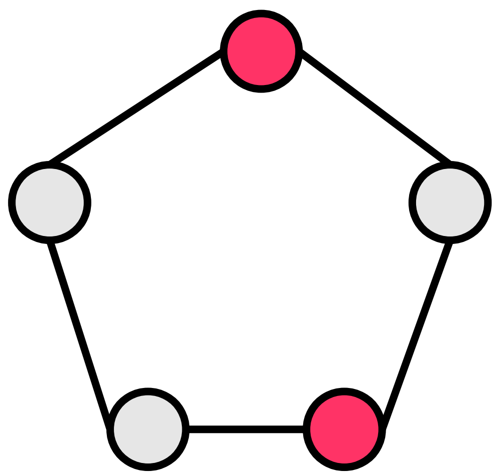
Let Γ be a compatibility structure, seen as a hypergraph. Let G be the graph obtained by connecting contextual questions. The <u>**maximum expectation values**</u> for exclusive answers are

$$\alpha(G) \leqslant \vartheta(G) \leqslant \alpha^{FP}(\Gamma)$$

for classical, quantum, and non-signaling theories, respectively; where $\alpha(G)$ is the independence number, $\vartheta(G)$ is the Lovász ϑ-function, and $\alpha^{FP}(\Gamma)$ is the fractional packing number.

Independence number $\alpha(G)$

Is the maximum number of mutually non-adjacent vertices in a graph *G*.

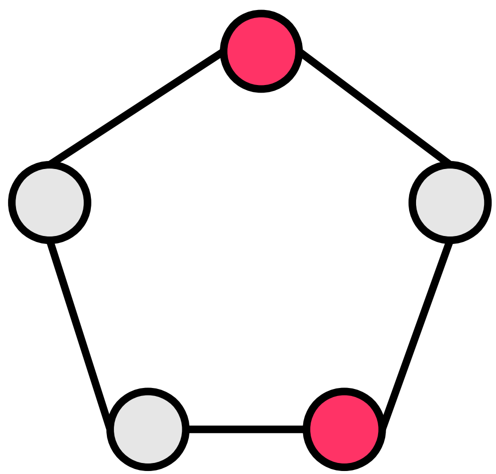


$\alpha(C_5) = 2$

NP-complete;
hard to approximate

Independence number $\alpha(G)$

Is the maximum number of mutually non-adjacent vertices in a graph G.



$\alpha(C_5) = 2$

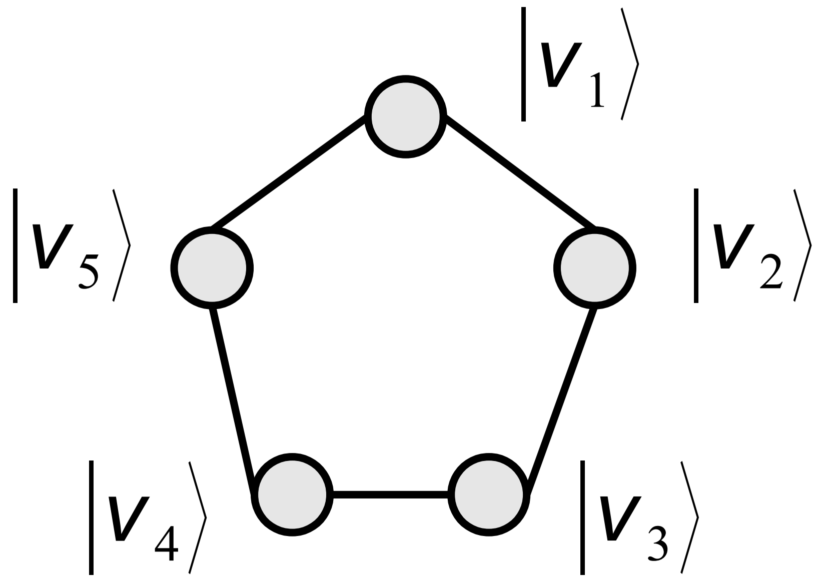
NP-complete;
hard to approximate

classical theories
give expectation ≤ 2

Lovász ϑ-function $\vartheta(G)$

An **orthogonal representation** of G is a set of unit vectors associated to the vertices such that two vectors are orthogonal if the corresponding vertices are adjacent:

$$\vartheta(G) := \max_{\text{orth. repr.}} \sum_{i=1}^{n} \left|\langle \psi | v_i \rangle\right|^2$$



$$\vartheta(C_5) = \sqrt{5}$$

semidefinite program*

*Lovász (1978)

Lovász ϑ-function $\vartheta(G)$

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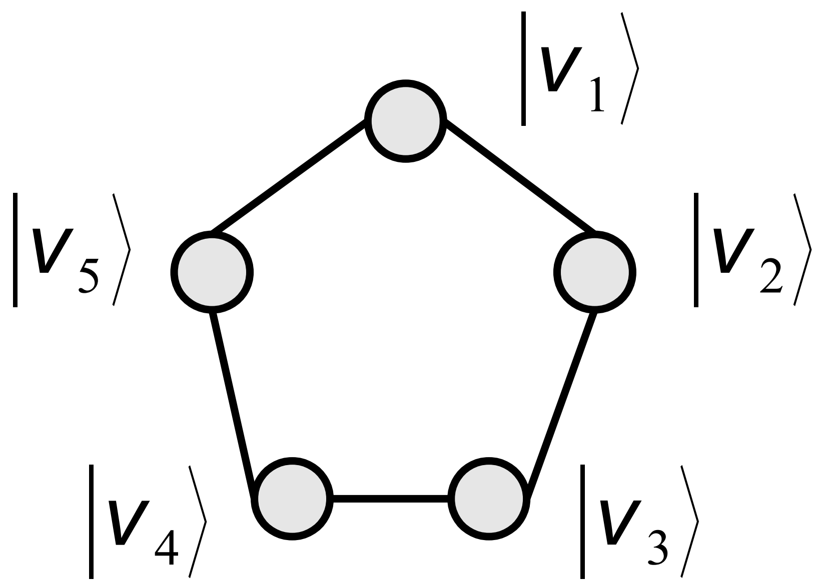


$$\vartheta(C_5) = \sqrt{5}$$

semidefinite program*

quantum theory
gives expectation ≤ $\sqrt{5}$

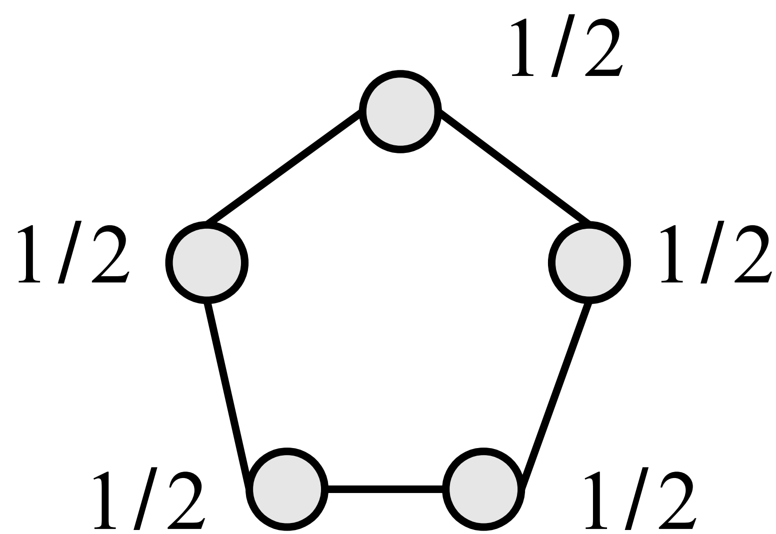
*Lovász (1978)

Fractional packing number $\alpha^{FP}(\Gamma)$

Let Γ be a compatibility structure, seen as a hypergraph:

$$\alpha^{FP}(\Gamma) = \max \sum_i w_i$$

$$s.t. \; \forall\, i \quad 0 \leqslant w_i \leqslant 1 \quad \text{and} \quad \forall \text{ context } \; C \in \Gamma, \quad \sum_{i \in C} w_i \leqslant 1$$



$\alpha^{FP}(C_5) = 5/2$

linear program

Fractional packing number $\alpha^{FP}(\Gamma)$

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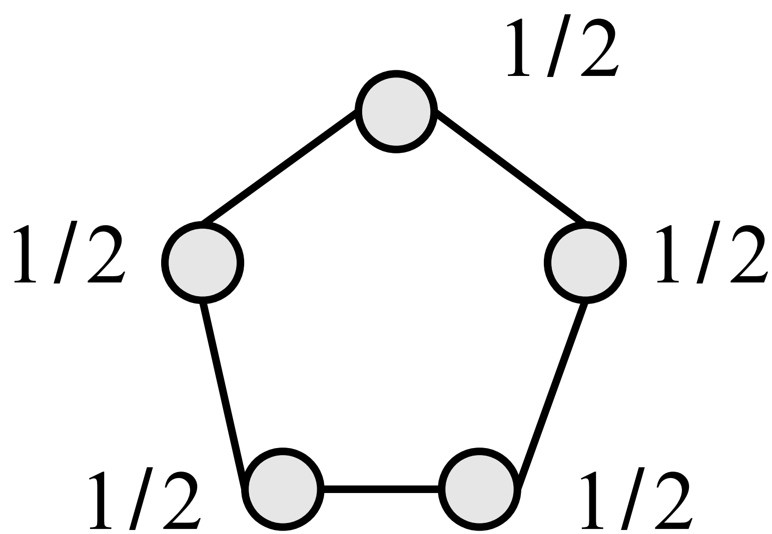


$$\alpha^{FP}(C_5) = 5/2$$

linear program

non-signaling theories
give expectation ≤ 2.5

Remark.

$\vartheta(C_5)$ is the max. violation of the **Klyachko-Can-Binicioglu-Shumovsky (KCBS) inequality***.
The inequality can be used to detect genuine quantum effects and it is the simplest non-contextual inequality violated by a qutrit (because the orthogonal representation has dimension 3).

*Klyachko *et al.* (2008)

Quantum mechanics as a "sandwich theory"*

Let $E_{\mathrm{C}} \subset E_{\mathrm{Q}} \subset E_{\mathrm{NS}}$ be the **convex sets** of the vectors realizing the expectations for classical, quantum, and non-signaling theories, respectively.

*cf. Knuth (1994)

Quantum mechanics as a "sandwich theory"*

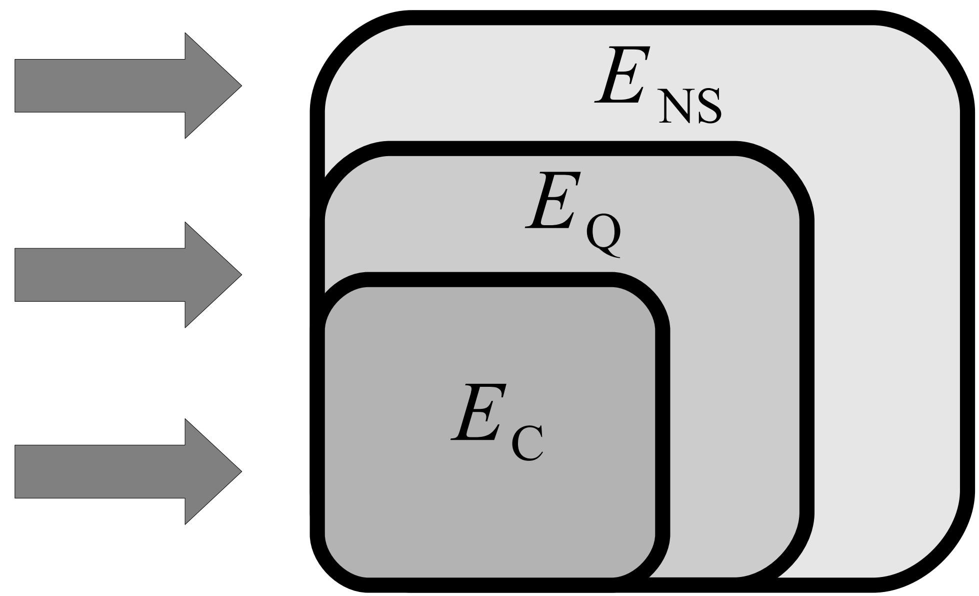
Let $E_{\mathrm{C}} \subset E_{\mathrm{Q}} \subset E_{\mathrm{NS}}$ be the **convex sets** of the vectors realizing the expectations for classical, quantum, and non-signaling theories, respectively.

Maxima

$\alpha^{FP}(\Gamma)$

$\vartheta(G)$

$\alpha(G)$



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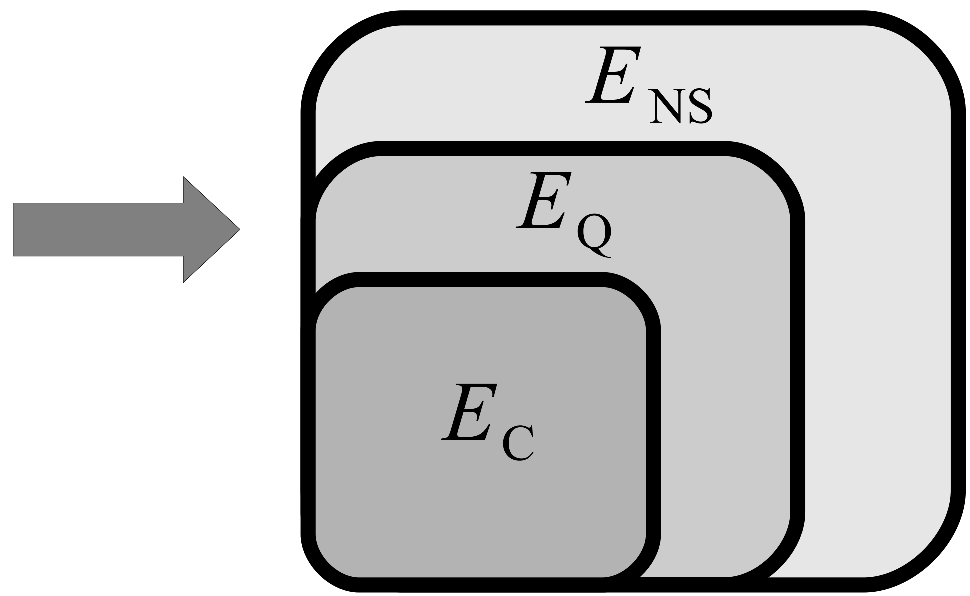
Quantum mechanics as a "sandwich theory"*

Let $E_{\mathrm{C}} \subset E_{\mathrm{Q}} \subset E_{\mathrm{NS}}$ be the **convex sets** of the vectors realizing the expectations for classical, quantum, and non-signaling theories, respectively.

membership in E_{Q} of a vector can be tested with a semidefinite program!



*cf. Knuth (1994)

Remark.

Standard result about the Lovász function can be then used to give the max. violation of known inequalities. For example, the max. violation for the *n*-cycle generalization of the KCBS inequality, recently computed in * is

$$\vartheta(C_n) = \frac{n\cos(\pi/n)}{(1+\cos(\pi/n))}$$

*Liang-Spekkens-Wiseman (2010)

Classicality and perfectness

A graph G is **perfect*** if $\alpha(H)=\vartheta(H)=\chi(\bar{H})$ for every induced subgraph H. So, perfect graphs are "the most classical ones". For a perfect graph

$$E_{\mathrm{C}}=E_{\mathrm{Q}}=E_{\mathrm{NS}}$$

Whenever $\alpha(G)<\vartheta(G)$ we have a difference between classical theories and quantum mechanics and a "state dependent" proof of the Kochen-Specker theorem**.

The KCBS inequality is based on C_5 which is the smallest non-perfect graph.

*Berge (1961); **where effects sum to unity

Two remarks

1. Many intractable problems are tractable for perfect graphs (*i.e.*, when classical and quantum theories coincide).

2. There are graphs s.t. $\alpha(G)=2$ and $\vartheta(G)=\Omega(n^{1/3})$ (*i.e.*, classical and quantum theories can have arbitrarily large separation)*.

*Koniagin (1981)

non-contextuality

↕

non-locality

Observation

Non-local experiments give compatibility structures: compatible questions are the local measurement.

Observation

Non-local experiments give compatibility structures: compatible questions are the local measurement.

Alice	**Bob**
settings $x \in X$	settings $y \in Y$
outcomes $a \in A$	outcomes $b \in B$

Compatibility graph for a non-local experiment

$$G = (V, E) \qquad V = A \times B \times X \times Y$$

$$\{abxy, a'b'x'y\} \in E \text{ iff}$$
$$(x = x' \wedge a \neq a') \vee (y = y' \wedge b \neq b')$$

[Γ is the hypergraph of all cliques* in G]

*complete subgraphs

Compatibility graph for a non-local experiment

$G=(V,E)$ $\quad\quad V=A\times B\times X\times Y$

$$\{abxy, a'b'x'y\}\in E \text{ iff}$$
$$(x=x'\wedge a\neq a')\vee(y=y'\wedge b\neq b')$$

⇕

exclusiveness;
compatibility:
the observables of
Alice and Bob **commute**.

A classification theorem for correlations

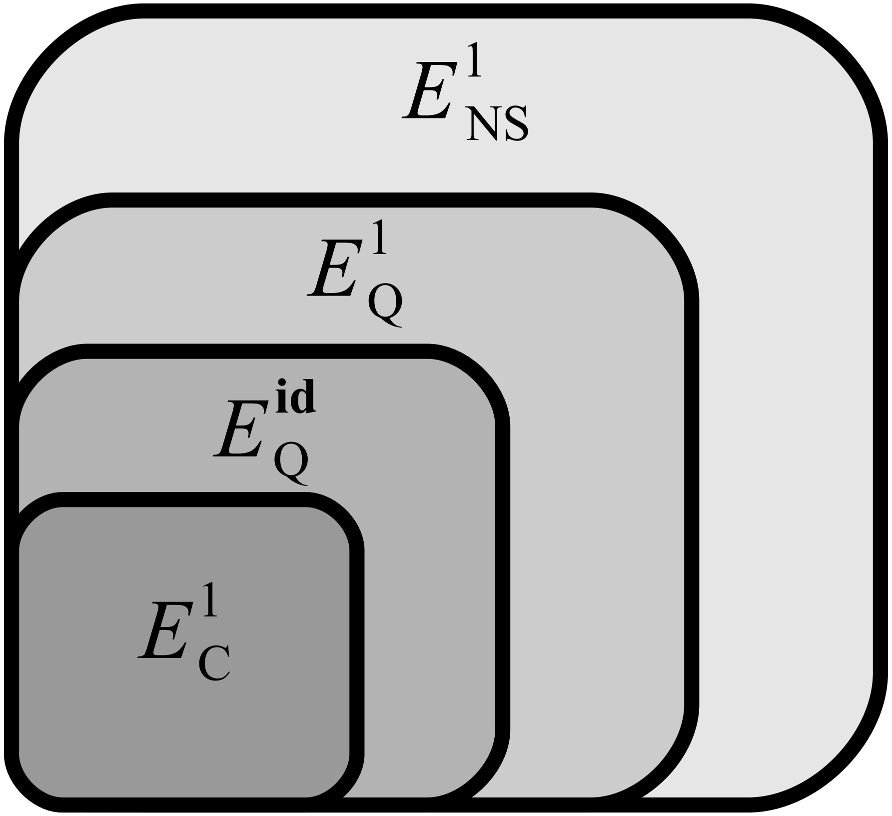
Let Γ be the compatibility hypergraph for a non-local experiment:

$$E^1_{\mathrm{C}}(\Gamma) \subset E^{\mathbf{id}}_{\mathrm{Q}}(\Gamma) \subset E^1_{\mathrm{Q}}(\Gamma) \subset E^1_{\mathrm{NS}}(\Gamma)$$

are the sets of correlations obtainable by local hidden variables, local quantum measurements on a bipartite state, *idem* but without completeness relation for the measurement, and non-signaling theories, respectively:

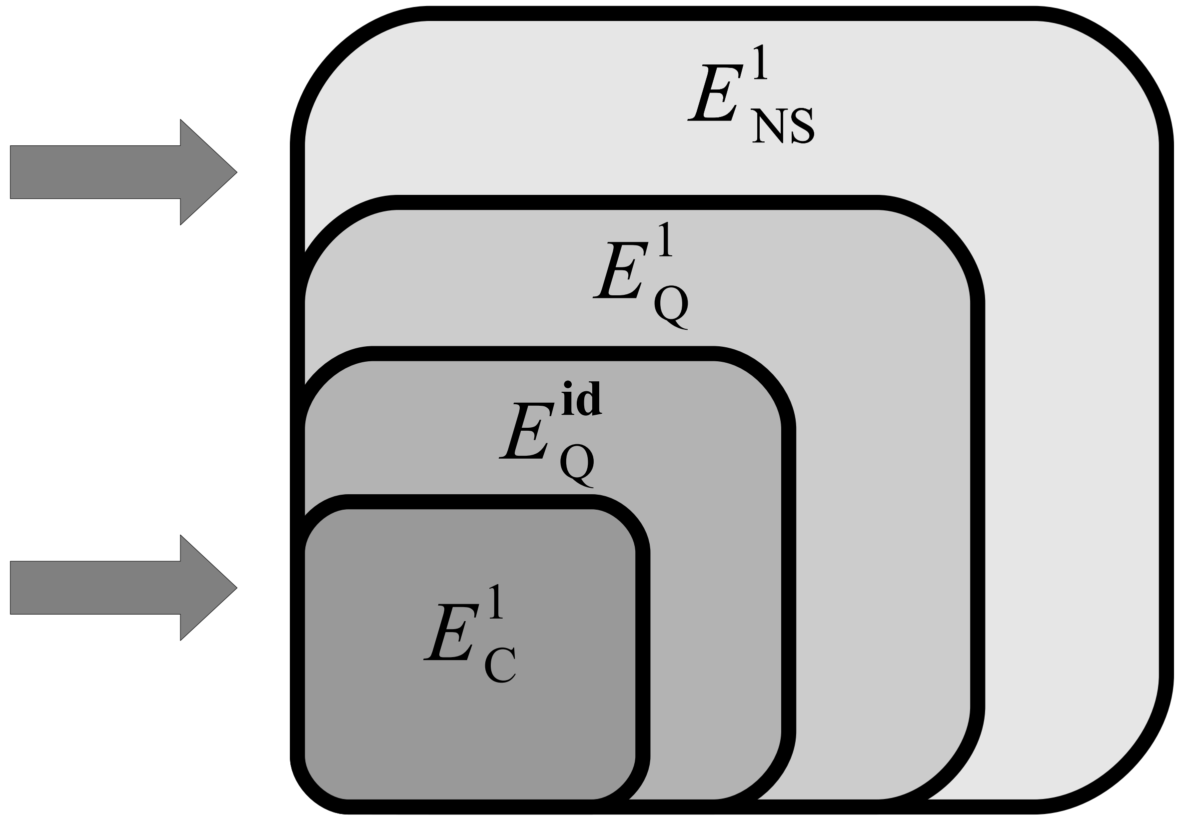
$$E^1_{\mathrm{X=C,Q,NS}}(\Gamma) := E_{\mathrm{X}}(\Gamma) \cap \{\vec{w} \ : \ \forall\, xy \ \sum_{w_{ab|xy}} w_{ab|xy} = 1\}$$

$$E^{\mathbf{id}}_{\mathrm{Q}}(\Gamma) := \{(w_{ab|xy})_{abxy} \ : \ \forall\, xy \ \sum_{ab} P_{ab|xy} = \mathbf{id}\}$$


E^1_{NS}
E^1_Q
$E^{\mathbf{id}}_Q$
E^1_C

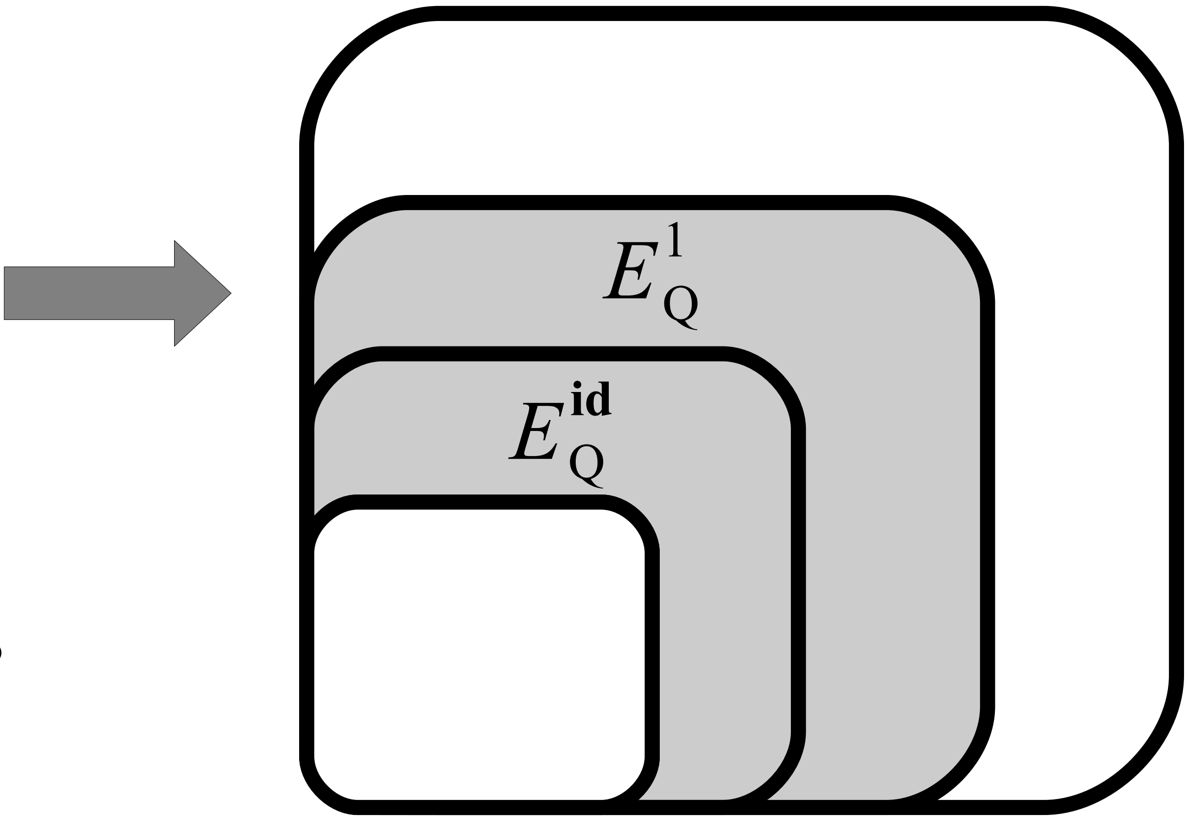
Fact

maximization over
E^1_{NS} and E^1_{C}
is equivalent to
maximization over
E_{NS} and E_{C}



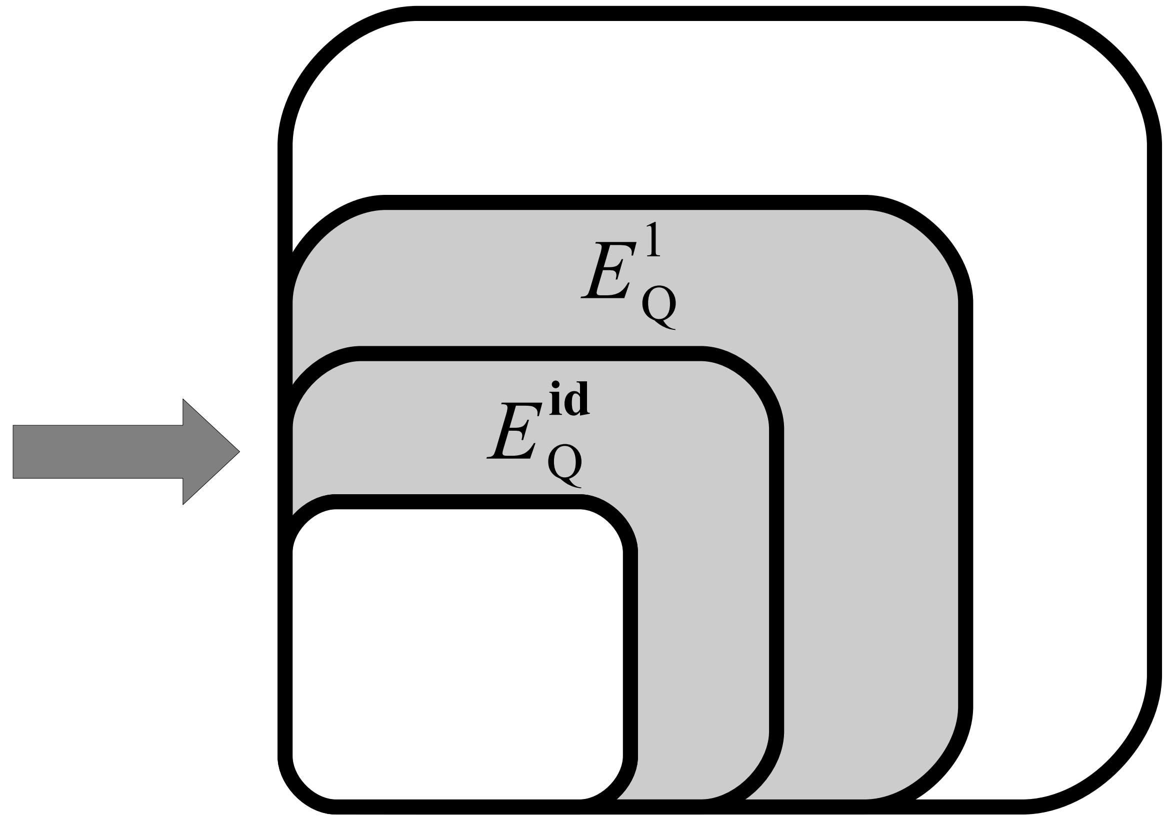
Fact

maxima via SDP are **efficient upper bounds** to maximum quantum violations



Fact

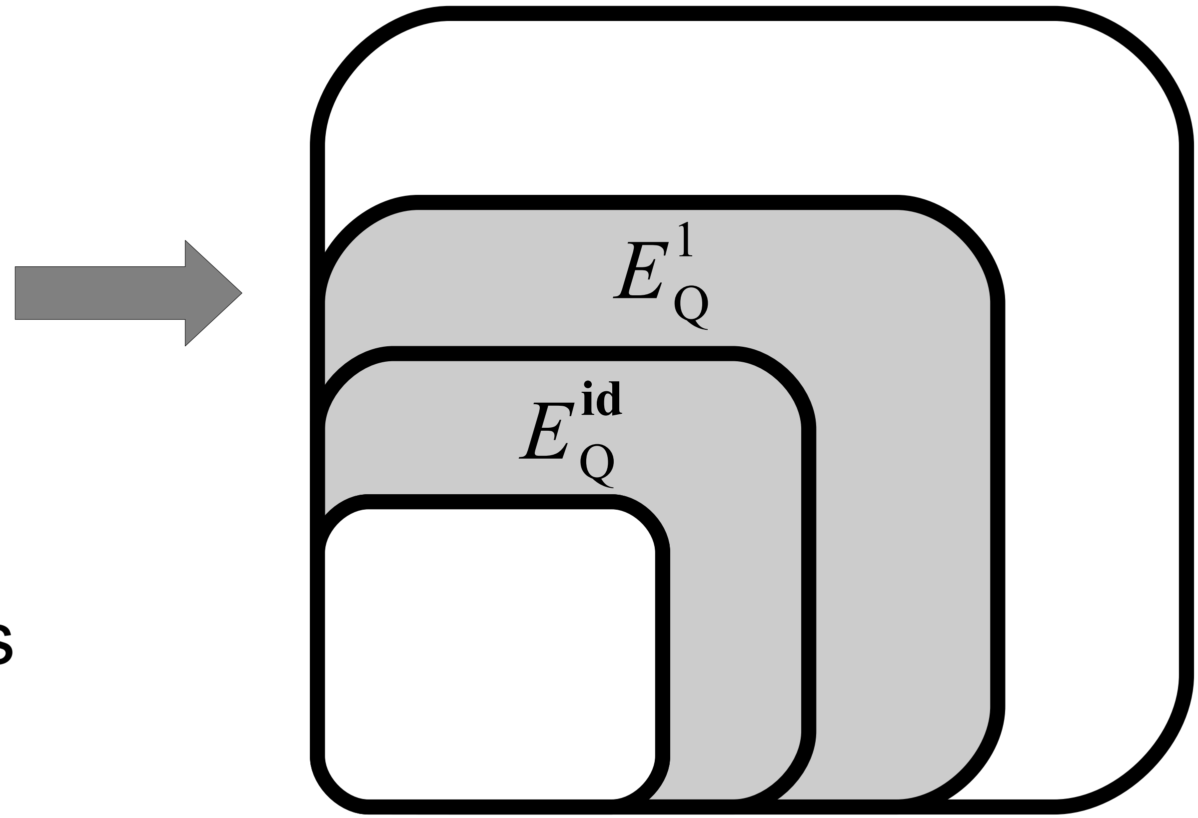
there is no efficient algorithm, unless the polynomial hierarchy collapses*



*Ito-Kobayashi-Matsumoto (2009)

Fact

maxima via SDP are **efficient upper bounds** to maximum quantum violations



Problem: how well E_Q^1 approximates $E_Q^{\mathbf{id}}$?

Clause-Horne-Shimony-Holt (CHSH) inequality*

*Clause-Horne-Shimony-Holt (1969)

CHSH inequality

settings and outcomes: $A = B = X = Y = \{0,1\}$

constraint: $$\sum_{w_{ab|xy} : x \cdot y = a \,\mathrm{XOR}\, b} w_{ab|xy}$$

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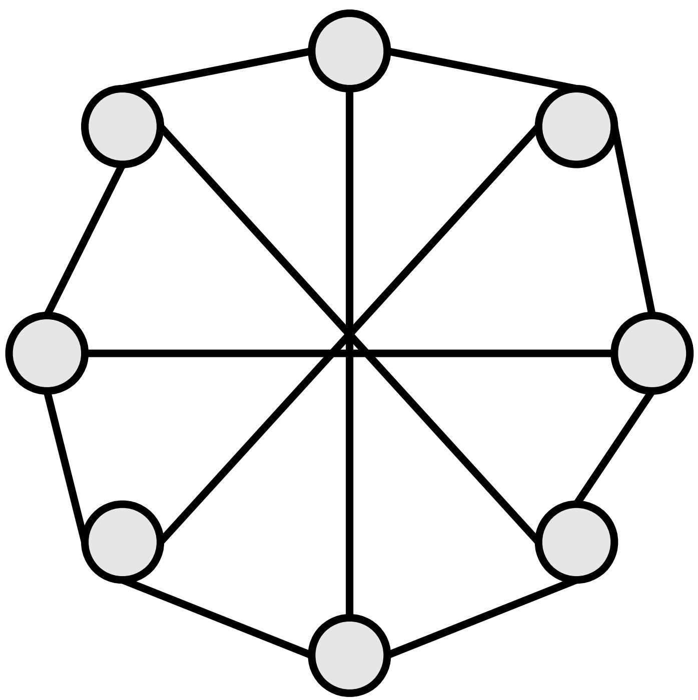

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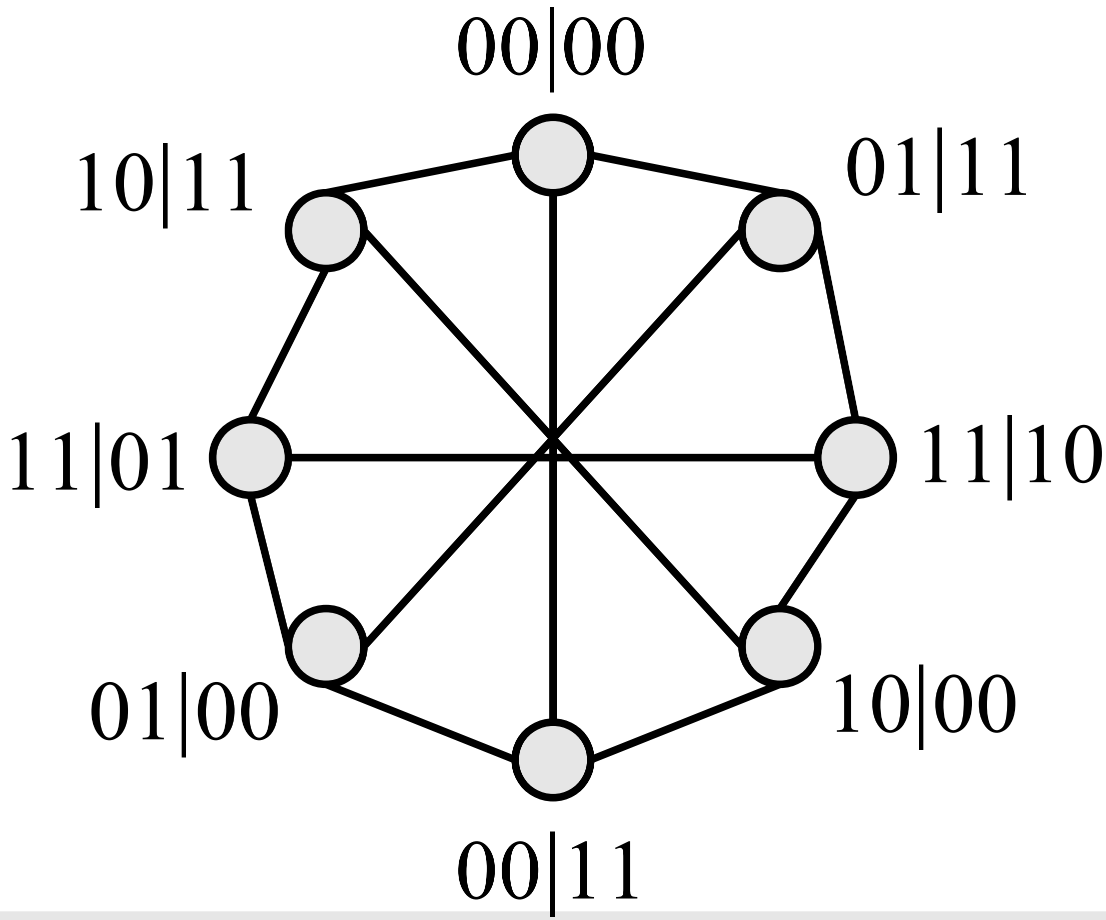

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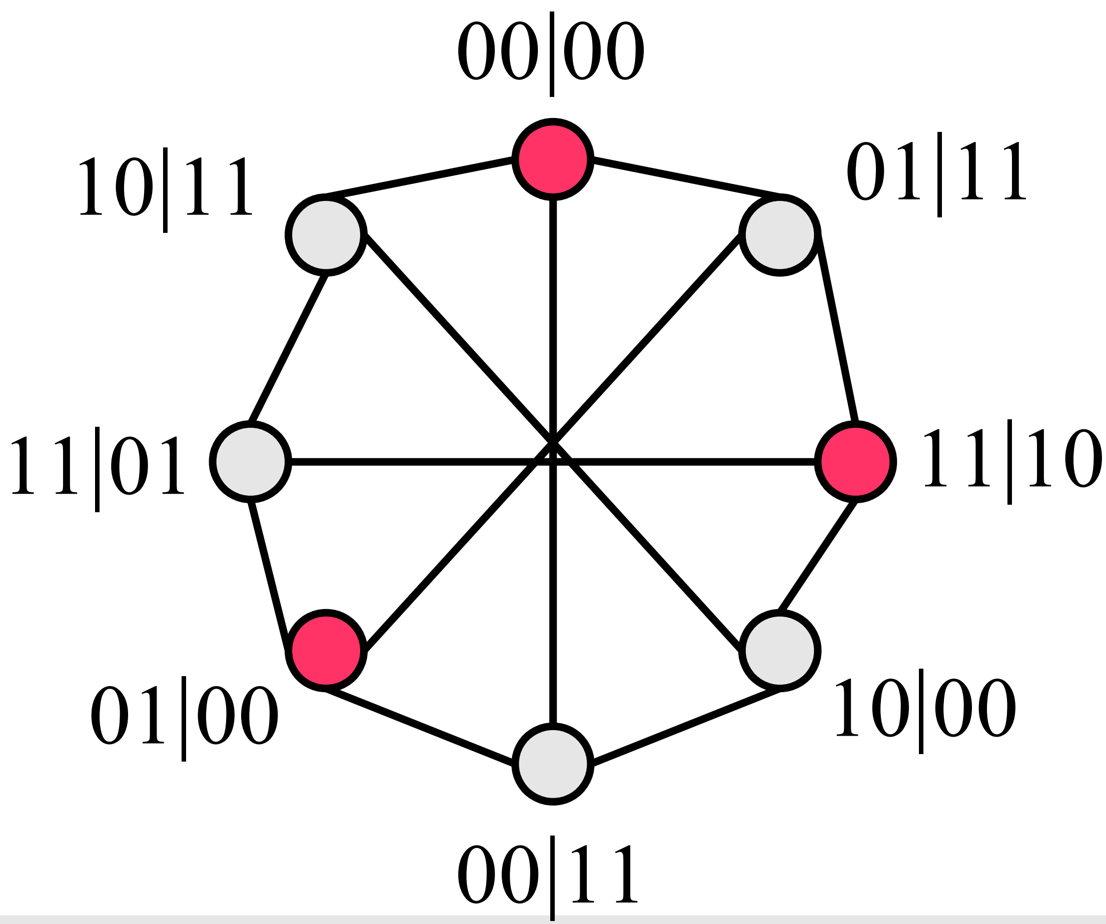

$\alpha(G) = 3$

classical max.

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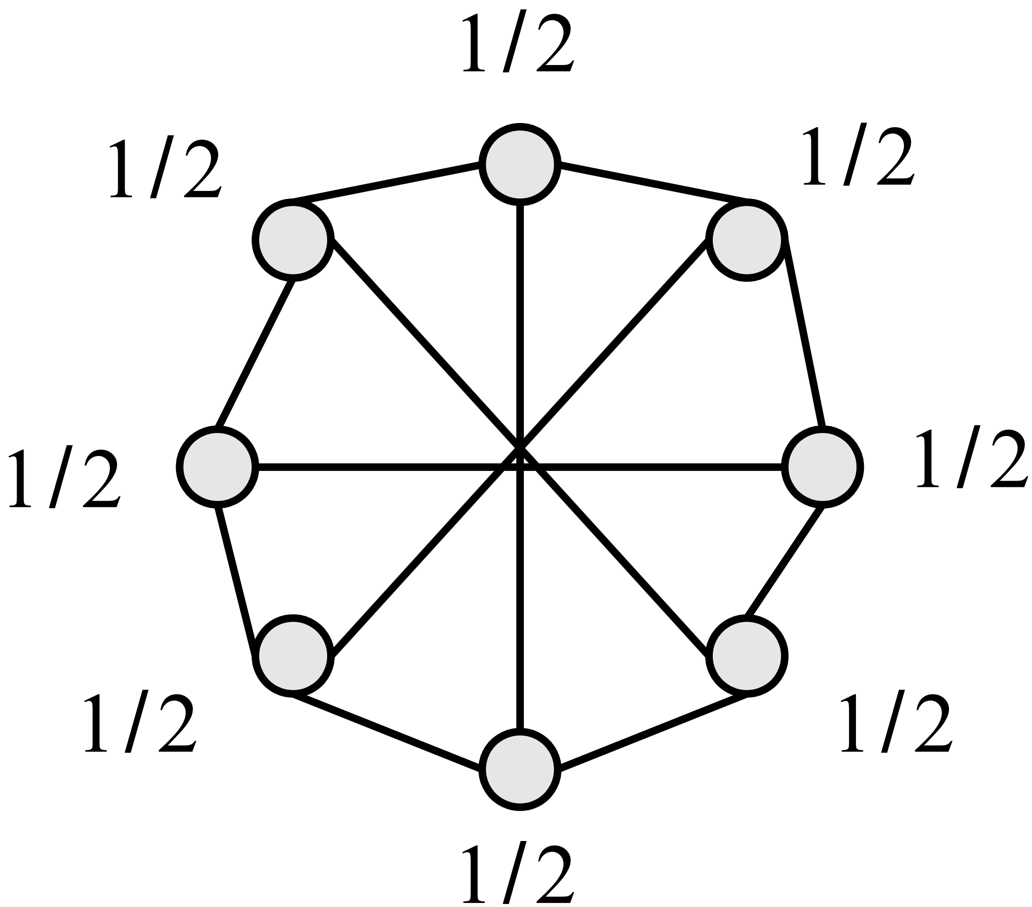

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$\alpha^{\mathrm{FP}}(\Gamma) = 4$

non-signaling max.

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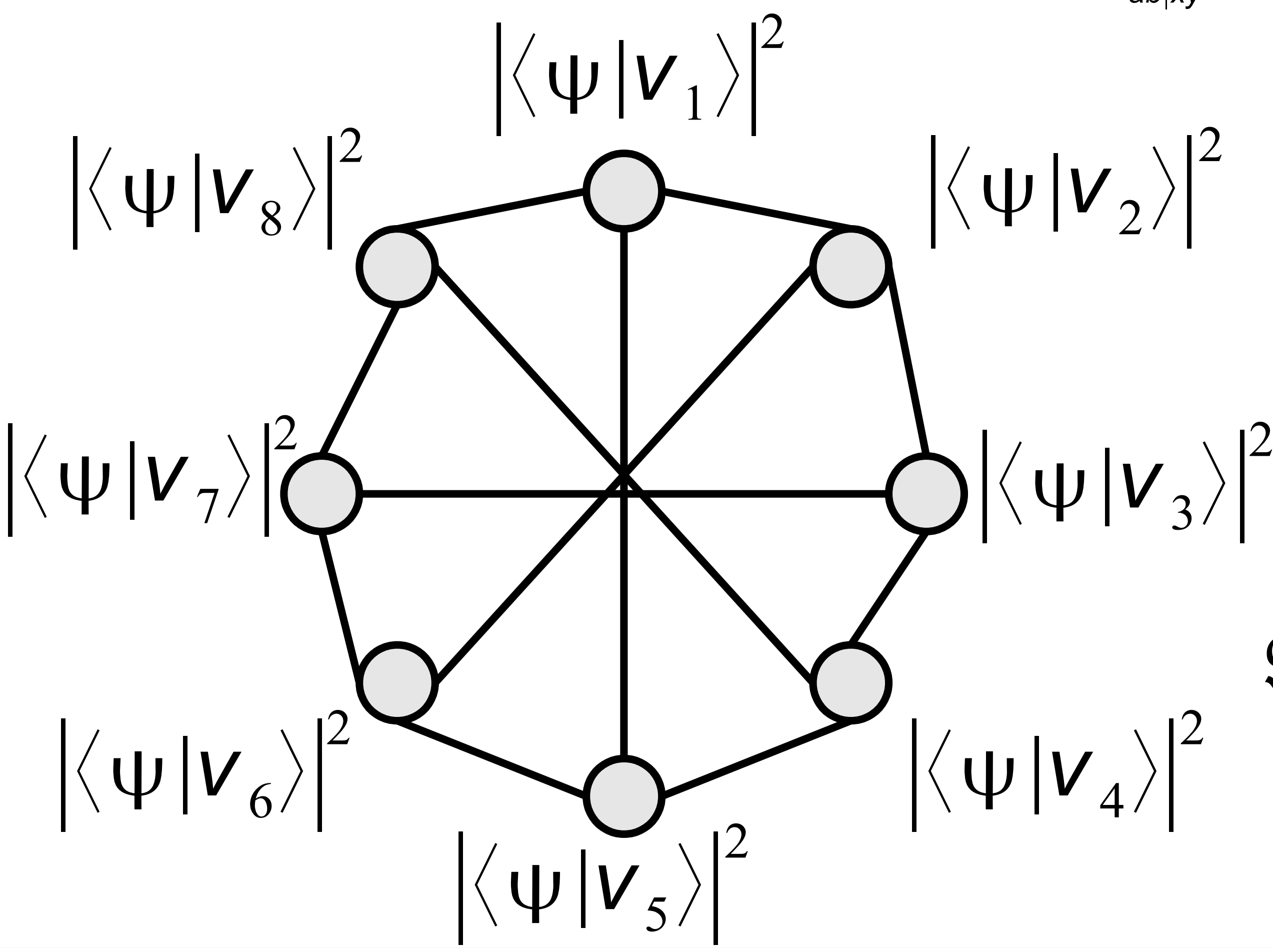

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$\vartheta(G) = 2 + \sqrt{2} \approx 3.4$

quantum max.

CHSH inequality

$\alpha(G)=3$

classical max.

$\alpha^{FP}(G)=4$

non-signaling max.

it attains the
Tsirelson bound

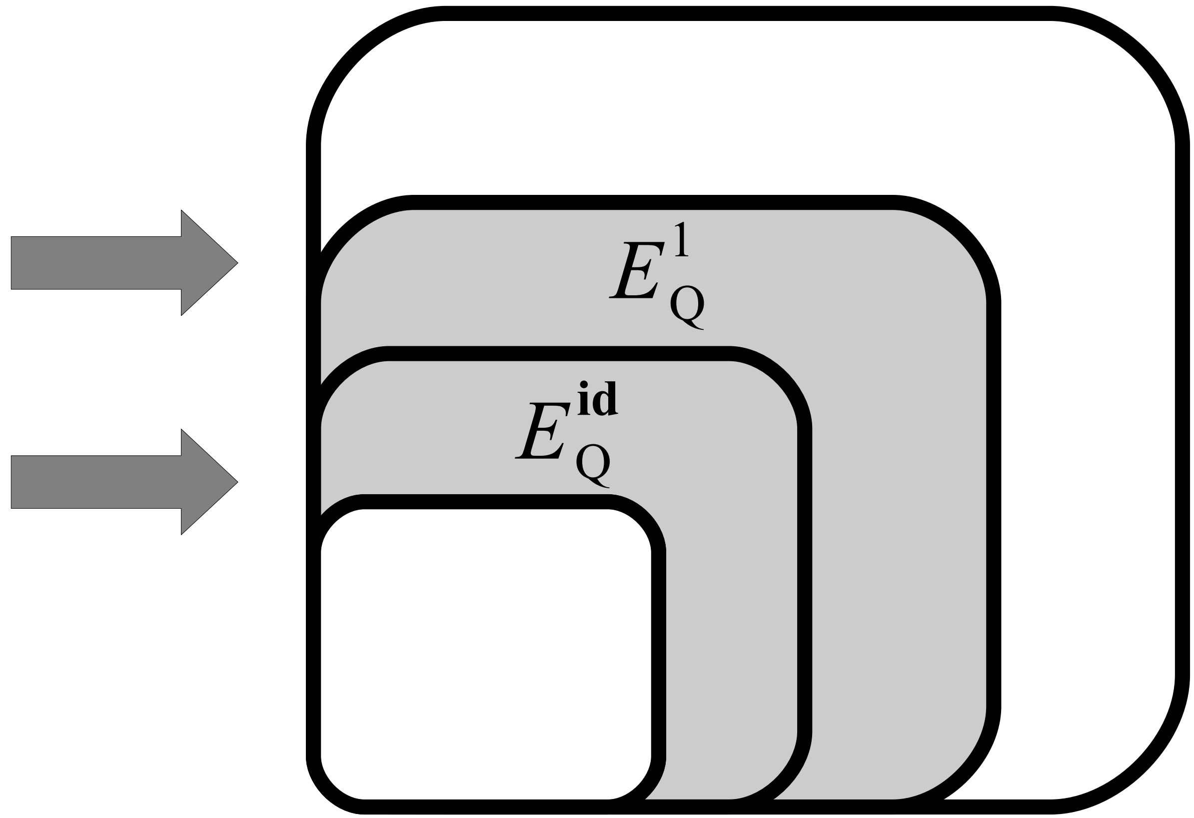
$\vartheta(G)=2+\sqrt{(2)}\approx 3.4$

quantum max.

Collins-Gisin inequality (I3322)*

max. 6.2514

max. 6.2508**



$$E_{\mathrm{Q}}^{\mathbf{id}} \subset E_{\mathrm{Q}}^{1}$$

*Collins-Gisin (2004); **Navascués-Acín-Pironio (2008)

3. Open problems:

1. theoretical:
   relations to Bell inequalities
2. applied:
   loophole-free experiments
3. a complexity perspective:
   degree of perfectness

"theoretical open problem"

Can any violation of a non-contextual inequality be converted into a (comparably large) violation of a Bell inequality?

"applied open problem"

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So far, forty years after Bell paper, all Bell experiments have loopholes: are graphs with a large separation between the independence number and the Lovász function good candidates for loophole-free experiments with inefficient detectors?

“complexity open problem”

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Perfect graphs have many efficient algorithms that in general are NP-hard. We have shown that compatibility structures from perfect graphs have coincident classical and quantum description. Can we define a notion of parametric complexity according to the classical-quantum gap?

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The Lovász function is fundamental in zero-error classical and quantum information theory*. Can we recast the non-contextuality framework into an information theoretic one?

*09:55, Fri, Matthews' talk; poster Duan-Severini-Winter

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