Block Weighing Matrices

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Abstract

We define a special type of weighing matrix called *block weighing matrices*. Motivated by questions arising in the context of optical quantum computing, we prove that innite families of anticirculant block weighing matrices can be obtained from generic weighing matrices. The classification problem is left open.

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1 Introduction

Quantum computing is a paradigm which promises to overcome classical computing in tasks ranging from algorithms for group theoretic problems to the efficient simulation of quantum and chemical many-body systems [5]. The main issue of quantum computing is the physical implementation beyond the actual theoretical models. Indeed, preparing a quantum state and controlling its evolution is a major challenge from the theoretical and engineering point of view. This is because the interaction between the system introduces noise which is difficult to avoid.

One of the most promising approaches for implementing quantum computation is the one-way model [6]. In this model, unitary evolution (*i.e.*, quantum evolution isolated from the environment) is substituted by a sequence of measurements on specific subsystems. On one side this changes the perspective about protecting the system from noise. On the other side, the initial state of the system needs to be a special entangled state, *i.e.*, a state with non-classical correlations. This physical resource is called a *cluster state* [1]. A method to generate efficiently continuous-variable cluster states has been proposed in [4].

This method for preparing continuous-variable cluster states requires, on grounds of experimental and theoretical tractability, the existence of certain families of matrices [2]. The required matrices turn out to be weighing matrices. However, the matrices to be employed in this context

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need to have extra constraints. In particular, the matrices need to be in Hankel form, meaning that the skew-diagonals (*i.e.*, in the Northeast direction) of the matrix are all constant; an additional constraint imposed on these matrices from the theoretical perspective is that the main diagonal is zero. When the implementation scheme presents further constraints, then one is naturally lead to consider block-Hankel matrices. In such matrices, the size of the blocks is equal to the number of degenerate degrees of freedom. The existence of block-Hankel matrices of different orders and weights is parallel to the implementability of specific schemes.

A classification of these matrices would be a useful step towards a deeper understanding of quantum computing implemented with optical cavities. Motivated by the above context and considerations, in the present note, we define *block weighing matrices* and study some basic families of those arising from weighing matrices. In general, a block weighing matrix can be constructed from Latin Squares, $m \times m$ arrays of m symbols such that each symbol occurs only once in each row and column. Specifically, we examine two subfamilies of block weighing matrices: Hankel and anticirculant. A Hankel matrix is *anticirculant* if each row (except the first row) is obtained by its previous row by a left cyclic shift.

The structure of this paper is as follows: In Section 2, we give the definition of a block weighing matrix and its elements, and in Section 3 we show how block weighing matrices are generated from Latin Squares and that each weighing matrix gives rise to a family of anticirculant and Hankel block weighing matrices.

2 Elements of Block Weighing Matrices

In this section, we introduce the notion of a block weighing matrix. We assume the reader is already familiar with the elementary notions of matrix theory. We shall start by defining some basic terms:

Definition 1 (Elements of a Block Weighing Matrix) *Let $\overline{P} = \{P_1, P_2, ..., P_d\}$ be a set of orthogonal projectors of dimension d such that $P_i = v_i v_i^T$, where v_i is a vector of norm 1, for every $i = 1, 2, ..., d$, and $\sum_{i=1}^{d} P_i = I_d$, where I_d is the identity matrix of dimension d. Such a set is said to be the* set of elements of a block weighing matrix.

As an example, let us consider the vectors $v_1 = (1,1)^T$ and $v_2 = (1,-1)^T$. Let $\overline{P} = \{P_1 = v_1 v_1^T, P_2 = v_2 v_2^T\}$. Then, $P_1 + P_2 = 2I_2$. Once normalized, the projectors in the set $\overline{P}$ can be seen as the elements of a block weighing matrix. In this paper, we will use unnormalized orthogonal projectors in our proofs as a matter of practicality and simplicity; as normalization is simply scalar multiplication of the projectors, this does not impact our on proofs. The following definition will guide our discussion.

Definition 2 (Block Weighing Matrix) *A matrix M is defined as a* block weighing matrix *$BW(n,d,k)$ of* order n, block size d, *and* block weight k *if its block entries are $B_{i,j} \in \{0_d, \overline{P}\}$, for 0_d a $d \times d$ zero matrix, each block row of B contains exactly k nonzero blocks, and the matrix is orthogonal up to normalization.*

Recall that an $n \times n$ matrix M is a *weighing matrix* $W(d,k)$ of *order* n and *weight* k if its entries are in the set $\{0, \pm 1\}$, there are exactly k nonzero entries in each row, and $MM^T = M^T M = kI_n$ [3]. It is interesting to note that any $W(d,k)$ weighing matrix can generate the elements of a block weighing matrix. Extending the previous example, let us write $M = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_1 \end{pmatrix}$, where

$P_1, P_2 \in \overline{P}$. The matrix M is a block weighing matrix $BW(4,2,2)$. Indeed, it turns out that $M = H_4$, where H_4 is the Hadamard matrix of order 4:

$$H_2 = \begin{pmatrix} P_1 & P_2 \\ P_2 & P_1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

The next simple statement gives a recipe for constructing a block weighing matrix from any weighing matrix $W(d,k)$. The result allows us to interpret some block weighing matrices as a special type of weighing matrix. The idea is straightforward: each row of a (normalized) weighing matrix is a unit vector with the same number of nonzero entries; all rows form an orthonormal basis.

Proposition 3 *Let M be a $W(d,k)$ weighing matrix of order d and weight k, and let us label the rows of this matrix as $R_1, R_2, ..., R_d$. Let us define $P_i = R_i^T R_i$. Then the set $\overline{P} = \{P_1, P_2, ..., P_d\}$ is a set of block weighing matrix elements.*

Proof. By definition, we can write $P_i P_j$ as $R_i^T R_i R_j^T R_j$. If $i = j$, then we know that $P_i P_j = R_i^T R_i R_i^T R_i$. So, as there are k nonzero elements in each row of M, we have that $R_i^T R_i = k$. Hence, $R_i^T R_i R_i^T R_i = R_i^T k R_i = k R_i^T R_i = k P_i$. If $i \neq j$, then $R_i^T R_i R_j^T R_j = R_i^T 0 R_j = 0_d$, as the rows of M are pairwise orthogonal. Hence, we have

$$P_i P_j = \begin{cases} 0_d, & i \neq j; \\ k P_i, & i = j. \end{cases}$$

Now we know that $\sum_{i=1}^{d} P_i = Z$, for some $d \times d$ matrix Z. Since the product of two distinct matrices of $\overline{P}$ is zero, we have that for any arbitrary matrix P_j from $\overline{P}$, $P_j \left(\sum_{i=1}^{d} P_i \right) = P_j^2 = k P_j = P_j Z$, so $P_j = P_j \frac{1}{k} Z$. A similar process demonstrates that $k P_j = \frac{1}{k} Z P_j$. Therefore, by the last two equations, $\frac{1}{k} Z = I_d$. Since $Z = k I_d$, we have $\sum_{i=1}^{d} P_i = k I_d$, and $\overline{P}$ satisfies Definition 2. ■

Notice that a block weighing matrix may not be a weighing matrix, because some of the projectors may have a different number of nonzero entries in each row. As an example, let us consider the vectors $v_1 = (1,1,0)^T$, $v_2 = (1,-1,0)^T$ and $v_3 = (1,-1,0)^T$, forming an orthonormal basis. Let $\overline{P} = \{P_1, P_2, P_3\}$. It follows that

$$M = \begin{pmatrix} P_1 & P_2 & P_3 \\ P_3 & P_1 & P_2 \\ P_2 & P_3 & P_1 \end{pmatrix}$$

is a block weighing matrix $BW(9,3,3)$.

3 Hankel and Anticirculant Block Weighing Matrices

In this section we demonstrate that $BW(n,d,k)$ can be constructed from $n/d \times n/d$ Latin Squares whose symbols are taken from the multiset $\{P_0, P_1, ..., 0, ..., 0\}$, for $P_i \in \overline{P}$ as defined in Proposition 3. From this, we can readily construct block weighing matrices of anticirculant and Hankel designs.

Remark 4 *By definition, an $n \times n$ Latin Square is constructed from a set of n different symbols. As Proposition 5 will show, so far as the nonzero elements of $\{P_0, P_1, ..., 0, ..., 0\}$ occur no more than once in any row or column, we will have a BWM. In keeping with the notion for Latin Squares, the zeros of $\{P_0, P_1, ..., 0, ..., 0\}$ will be treated as different symbols of our alphabet for exploiting the properties of Latin Squares; a similar abuse of the definition will be used in Corollary 6.*

Proposition 5 *Let $\overline{P}$ be a set of matrices generated as in Proposition 3 by a $W(d, k)$ weighing matrix. Suppose we have a $n/d \times n/d$ Latin Square L whose symbols are from the multiset $X = \{P_0, P_1, ..., P_d, 0, ...0\}$. Then L is a $BW(n, d, k)$ block weighing matrix.*

Proof. Suppose we have a matrix specified as above. Using L to both denote the Latin Square and the corresponding matrix, we have that by definition each block element of X occurs only once in each block row and block column of L. Let us consider $L \cdot L^T$ and the calculation of a specific block element. We may write this calculation as $A \cdot B = \sum_{i=0}^{i=n/d} A_i B_i$ where A is a block row of L and B is a block column of L^T, which is to say, a block row of L. As L is a Latin Square, it is quite clear by Proposition 3 that $L \cdot L^T = kI_d$ as each symbol of X occurs only once in each block row and block column. The main block diagonal elements are such that $A_i = Bi \ \forall\ i$ and the off block diagonal elements are such that $A_i \neq B_i \ \forall\ i$. ■

As anticirculant Latin Squares are specific subsets of these Latin Squares, we have the following corollary:

Corollary 6 *Let $X = \{P_0, P_1, ...P_d, 0, ...0\}$ be a multiset with n/d elements. Suppose we have a $n/d \times n/d$ Latin Square L of anticirculant design whose elements are taken from X. Then the matrix L is an anticirculant block weighing matrix and is denoted $ABW(n, d, k)$, in keeping with the notation for Block Weighing Matrices.*

The construction of Hankel Block Weighing Matrices are specifically motivated by [4]. Hankel Matrices in general require only that the skew diagonals be constant; however, Hankel Block Weighing Matrices will necessarily require the corresponding skew-diagonals to be either equal or additive inverses of each other. For example,

$$\begin{pmatrix} a_0 & a_1 & \dots & a_{d-1} \\ a_1 & a_2 & \dots & -a_0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{d-1} & -a_0 & \dots & -a_{d-2} \end{pmatrix}$$

is an example of a valid $BW(d^2, d, k)$, assuming $a_i \in \overline{P}$ for some valid set of elements of a block weighing matrix. First, we must prove two straightforward lemmas.

Lemma 7 *Let M be a $W(d, k)$ weighing matrix with rows $R_1, R_2, ..., R_d$. Let $\overline{P} = \{P_1, P_2, ..., P_d\}$ be a set of $d \times d$ matrices defined by $P_i = R_i^T R_i$. Suppose that we have $\sum_{i=0}^{l} X_i = kI_d$, for some blocks $X = \{X_1, X_2, ..., X_l\}$, such that $l \geq d$ and each one of those matrices is either part of $\overline{P}$ or 0. Then each element of $\overline{P}$ appears exactly once in $X_1, X_2, ..., X_l$, and these are the only nonzero elements in X.*

Proof. By Proposition 3, we have $\sum_{i=0}^{l} X_i = \sum_{i=0}^{d} P_i$. We can write $\sum_{i=0}^{l} X_i = \sum_{i=0}^{d} P_i + \sum_{i=0}^{d'} T_i - \sum_{i=0}^{d''} S_i + \sum_{i=0}^{d'''} Z_i$, where (abusing notation) T is the multiset of repetitions of elements of $\overline{P}$, S is the set of elements of $\overline{P}$ that are not in X, and Z is the set of zero blocks. Then we have that

$$\begin{gathered} \sum_{i=0}^{d} P_i + \sum_{i=0}^{d'} T_i - \sum_{i=0}^{d''} S_i = kI_d \\ \sum_{i=0}^{d} P_i = kI_d - \sum_{i=0}^{d'} T_i + \sum_{i=0}^{d''} S_i \end{gathered}$$

Equality can only hold if $\sum_{i=0}^{d'} T_i = \sum_{i=0}^{d''} S_i$, which implies that there are no repetitions of the elements of $\overline{P}$ in X nor any elements in $\overline{P}$ missing in X. Thus $X = \overline{P}$. ■

Lemma 8 *Let M be a $W(d,k)$ weighing matrix with rows $R_1, R_2, ..., R_d$. Let $\overline{P} = \{P_1, P_2, ..., P_d\}$ be a set of $d \times d$ matrices defined by $P_i = R_i^T R_i$. Then $P_i^2 \neq P_j^2$ and $(-P_i)^2 \neq (-P_j)^2$ for all $i \neq j$.*

Proof. It is sufficient to observe that

$$(-P_i)^2 = P_i^2 = R_i^T R_i R_i^T R_i = k R_i^T R_i = k P_i,$$

for every $i = 1, 2, ..., d$. From this, $P_i \neq P_j$. ■

Theorem 9 *Let $\overline{P}$ be a set of matrices generated as in Proposition 3 by a $W(d,k)$ weighing matrix. Suppose we have a block matrix*

$$B = \begin{pmatrix} D_0 & D_1 & \cdots & D_{n-1} \\ D_1 & D_2 & \cdots & Q_0 \\ \vdots & \vdots & \ddots & \vdots \\ D_{n-1} & Q_0 & \cdots & Q_{n-2} \end{pmatrix},$$

such that each block is either a zero block or an element of $\overline{P}$. Moreover, each row contains exactly k nonzero blocks. Then B is a $BW(nd,d,k)$ block weighing matrix if and only if each element of $\overline{P}$ is in the first row and $Q_i = \pm D_i$, for $i = 1, 2, ..., n-2$.

Proof. Let us assume that each element of $\overline{P}$ is in the first row and that $Q_i = \pm D_i$, for $i = 1, 2, ..., n-2$. Since B is of Hankel form, it is also symmetric. Thus the ij-th block of BB^T is $\sum_{i=0}^{n-1+i} D_i D_{i+j (\mathrm{mod}\, n)}$. By Proposition 3, $BB^T = k^2 I_{nd}$, and B is $BW(nd,d,k)$ block weighing matrix. Now let us assume that B is a matrix of this kind. Consequently, the matrix:

$$\begin{pmatrix} \sum_{i=0}^{n-1} D_i^2 & \sum_{i=0}^{n-2} D_i D_{i+1} + D_{n-1} Q_0 & \cdots & D_0 D_{n-1} + \sum_{i=1}^{n-1} D_i Q_{i-1} \\ \sum_{i=1}^{n-2} D_i D_{i+1} + Q_0 D_{n-1} & \sum_{i=1}^{n-1} D_i^2 + Q_0^2 & \cdots & D_0 D_{n-1} + \sum_{i=2}^{n-1} D_i Q_{i-2} + Q_0 Q_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ D_{n-1} D_0 + \sum_{j=0}^{n-2} Q_i D_{j+1} & D_{n-1} D_1 + \sum_{j=0}^{n-3} Q_i D_{j+2} + Q_{n-2} Q_0 & \cdots & \sum_{i=n-1}^{n-1} D_i^2 + \sum_{j=0}^{n-2} Q_j^2 \end{pmatrix}$$

is equal to $k^2 I_d$. Since the elements of B are either 0_d or from $\overline{P}$, we know that

$$D_i D_j = D_j D_i = Q_i D_j = Q_i Q_j = 0_d.$$

Along the main diagonal we have

$$\sum_{i=0}^{n-1} D_i^2 = k^2 I_d, \sum_{i=1}^{n-1} D_i^2 + Q_0^2, ..., \sum_{i=n-1}^{n-1} D_i^2 + \sum_{j=0}^{n-2} Q_j^2 = k^2 I_d.$$

So, by Lemma 7 and Lemma 8, all the matrices from these equations are either elements of $\overline{P}$ or zero matrices. Moreover, we know that the squares of elements from the set

$$\{D_0, D_1, ..., D_{n-1}\}, \{D_1, D_2, ..., D_{n-1}, Q_0\}, ..., \{D_{n-1}, Q_0, Q_1, ..., Q_{n-2}\}$$

must be equal to all the squares of elements of $\overline{P}$ exactly once. So, each set contains a linear combination of elements of $\overline{P}$. Thus, $Q_i = \pm D_i$, for $i = 1, 2, ..., d$. This concludes the proof. ■

Theorem 9 demonstrates that valid Hankel Block Weighing Matrices (denoted $HBW(n, d, k)$ to distinguish them from more general $BW(n, d, k)$) are very closely related to anticirculant block weighing matrices; indeed, Theorem 8 shows that Hankel BWM exist *only* if there is a $\pm D_i$ relation between the skew antidiagonals. Moreover, Theorem 9 helps greatly reduce the number of possible matrices and designs that are implementable. Our families of Hankel block weighing matrices are extensions of families of anticirculant block weighing matrices, since the antidiagonals are either equivalent or additive inverses of each other. The next statement follows on the basis of the above observations.

Corollary 10 *Let $W(d, k)$ be a weighing matrix. Then there is an infinite family of $HBW(nd, d, k)$ Hankel block weighing matrices for all $n \geq d$. Similarly, there is an infinite family of $ABW(nd, d, k)$ for all $n \geq d$.*

4 Conclusions

Corollary 10 is a useful statement because it gives a way to construct Hankel block weighing matrices from known weighing matrices. As we mentioned in the introduction, the existence of block Hankel weighing matrices is parallel to the implementability of specific schemes for optical quantum computing. The role of this observation is then to indicate a direction for the implementability of such schemes. Additionally, by considering a new type of weighing matrix, we propose space for a more refined classification of combinatorial design. The main open problem is now to determine if there exist anticirculant and Hankel block weighing matrices $ABW(n, d, k)$ and $HBW(n, d, k)$ that can not be obtained from weighing matrices of smaller orders, at least as it was described here.

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