Planetary Magnetospheres: Homework Problems

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1. In classical electromagnetic theory, the *magnetic moment* μ_L associated with a circular current 'loop' of radius R which carries a current I is given by the product of current and loop area:

$$\mu_L = I\,\pi R^2.$$

Apply this definition to the current carried by a particle of charge q and mass m gyrating in a single plane about a magnetic field of strength B. The particle thus moves on a circular orbit with speed $v_\perp$ and radius $r_g = m\,v_\perp/(qB)$. Show that the magnetic moment associated with the current represented by the particle's motion is equal to the first adiabatic invariant discussed in lectures, i.e. $\mu = W_\perp/B$, the ratio of gyrational kinetic energy to field strength.

Solution

Current is charge per unit time which passes a fixed point. For the particle, this may be written $I = q/T$, where T is the gyroperiod, i.e. $I = q^2B/(2\pi m)$. The area of the orbital circle is $A = \pi r_g^2 = \pi m^2 v_\perp^2/(q^2B^2)$. Hence $IA = \frac{1}{2}mv_\perp^2/B = W_\perp/B$.

2. For an ideal collisionless plasma of bulk velocity $\boldsymbol{u}$, Ohm's Law reduces to

$$\boldsymbol{E} = -\boldsymbol{u} \times \boldsymbol{B},$$

where $\boldsymbol{E}$ is the *convective* electric field. Show that the velocity component perpendicular to $\boldsymbol{B}$ is given by $\boldsymbol{u}_\perp = \boldsymbol{E} \times \boldsymbol{B}/B^2$.

Solution

Using the given definition of $\boldsymbol{E}$, we may write $\boldsymbol{E} \times \boldsymbol{B}/B^2 = (-\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B}/B^2$.

Now, $(-\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B}/B^2 = (B^2\boldsymbol{u} - (\boldsymbol{B} \cdot \boldsymbol{u})\boldsymbol{B})/B^2$.

If we define a unit vector $\boldsymbol{b} = \boldsymbol{B}/B$, we have $\boldsymbol{E} \times \boldsymbol{B}/B^2 = \boldsymbol{u} - (\boldsymbol{b} \cdot \boldsymbol{u})\,\boldsymbol{b} = \boldsymbol{u} - \boldsymbol{u}_{||} = \boldsymbol{u}_\perp$.

3. Following on from Question 2, a general plasma flow $\boldsymbol{u}$ is sometimes described by its corresponding pattern of convective electric field $\boldsymbol{E}$. If $\boldsymbol{E}$ can be described as the gradient of a scalar potential through $\boldsymbol{E} = -\nabla\phi_E$, then we have $\boldsymbol{u}_\perp = -\nabla\phi_E \times \boldsymbol{B}/B^2$.

Assume for simplicity that $\boldsymbol{u}_{||} = 0$.

Consider plasma motion in a 'magnetospheric equatorial' plane which contains the Earth-Sun line and is perpendicular to the Earth's magnetic dipole axis. Explain why the 'streamlines' of the plasma flow in this plane (curves which have a local tangent vector parallel to $\boldsymbol{u}$) are also curves of constant ϕ_E (i.e. *equipotential* curves).

In this equatorial plane, we may write ϕ_E as the sum of two terms:

$$\phi_E = \phi_{CR} + \phi_{CONV}.$$

The first term is the corotation potential and dominates close to the planet. It is given by:

$$\phi_{CR} = -\Omega_E\,B_E\,R_E^3/r,$$

where Ω_E is the Earth's angular velocity of rotation, B_E is the equatorial field strength at the Earth's surface, R_E is the Earth's radius and r is radial distance from the planet's centre.

The second term is the convection potential and describes sunward flows (associated with magnetotail reconnection) which carry plasma from the magnetotail towards the dayside:

$$\phi_{CONV} = -E_o\,y,$$

where E_o is the convection electric field (assumed constant) and y is the Cartesian coordinate measured along an axis (lying in the equatorial plane) which passes through the Earth's centre (the origin) and is perpendicular to the upstream solar wind direction (solar wind flows along the negative x direction). y is positive towards dusk.

There is a 'stagnation' point in the flow, lying on the positive y axis, whose location may be estimated as the point where the magnitudes of the two potential terms are equal. Show that the radial distance of the stagnation point is given (in units of Earth radii) by:

$$r_{sp}/R_E = (\Omega_E \, B_E \, R_E/E_O)^{1/2}$$

Using reasonable values for the Earth parameters, and a value $E_O = 1\,\mathrm{mV/m}$, calculate r_{sp}/R_E for the Earth's magnetosphere. How does variability in E_O affect this distance ?

For Jupiter, the planet's very strong field, size and rotation rate cause r_{sp} to lie *outside* the actual magnetosphere - what is the physical meaning of this result?

Solution Setting the magnitudes of ϕ_{CONV} and ϕ_{CR} to be equal, and using the fact that the radial distance r is equal to y for a point on the positive y axis, we obtain:

$$\Omega_E \, B_E \, R_E^3/r = E_O r \rightarrow (r/R_E) = \sqrt{\Omega_E \, B_E \, R_E/E_O}$$

Using the E_O value given (and transforming to MKS units), a rotation period of 24 hours for the Earth, a radius of 6370 km for the Earth, and $B_E = 3 \times 10^{-5}$ T, we obtain:

$$\begin{aligned}(r_{sp}/R_E) &= \sqrt{\Omega_E \, B_E \, R_E/E_O} \\ &= \sqrt{(2\pi/(24 \times 3600)) \times 3 \times 10^{-5} \times 6730 \times 10^3/10^{-3}} = 3.83\end{aligned}$$

An increase in E_O represents a stronger flow associated with the Dungey cycle, and a consequently smaller stagnation distance, which approximately represents the transition distance from sunward flow in the outer magnetosphere to corotational flow in the plasmasphere.

Jupiter's stagnation point lying outside its magnetosphere means that the dayside equatorial magnetosphere of Jupiter is dominated by rotational flows (more correctly, (sub)corotational with respect to the planet - see the lecture notes).

4. The magnetic field strength B due to the Earth's dipole field may be expressed as:

$$B = (B_E R_E^3/r^3)\,(3\cos^2\theta + 1)^{1/2}, \tag{1}$$

where B_E is the equatorial field strength at the Earth's surface, R_E is the Earth's radius and r is radial distance from the planet's centre. θ denotes magnetic colatitude (the magnetic equator is defined by $\theta = \pi/2$).

The following formula is for the pitch angle α_c associated with the *loss cone* at a point P where the field strength is B_P:

$$\sin^2\alpha_c = B_P/B_S, \tag{2}$$

where B_S is the magnetic field at the surface of the planet which is magnetically connected to the point P along the same field line.

Calculate the value α_c as a function of distance for locations in the magnetic equatorial plane, using the dipole approximation. You may find the following formula for the shape of a dipole magnetic field line useful:

$$r = LR_E \, \sin^2\theta, \tag{3}$$

where LR_E is the equatorial crossing distance of the field line.

Solution For any magnetic equatorial point at distance LR_E, a dipole field line passing through that point will intersect the Earth's surface at a colatitude θ_i given by:

$$R_E = LR_E \sin^2\theta_i$$
$$\rightarrow \sin\theta_i = \sqrt{1/L}$$
$$\rightarrow \cos\theta_i = \pm\sqrt{(L-1)/L} \tag{4}$$

Hence the magnetic field magnitude B_S is given by:

$$B_S = (B_E R_E^3/R_E^3)\,(3\cos^2\theta_i + 1)^{1/2} = B_E\,(3(1-1/L)+1)^{1/2}. \tag{5}$$

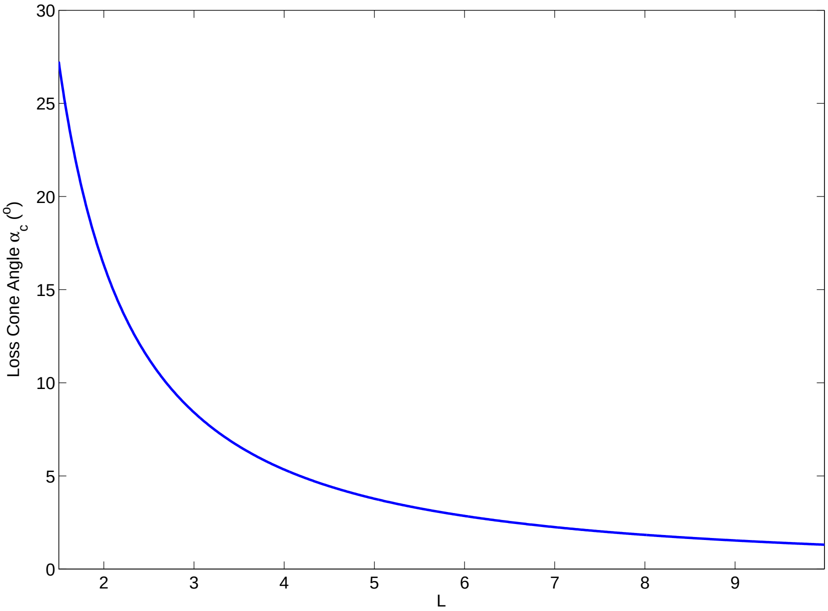
We can also evaluate the dipole formula at $\theta = \pi/2$, $r = LR_E$ to obtain B_P:

$$B_P = B_E/L^3. \tag{6}$$

It follows that:

$$\sin^2\alpha_c = B_P/B_S = L^{-3}\,(3(1-1/L)+1)^{-1/2} \tag{7}$$

Using this formula to evaluate $\sin^2\alpha_c$, hence α_c, as a function of L, we obtain the following plot:



5. The magnetic signatures of interchange observed by *Galileo* in Jupiter's magnetosphere indicate that the inward-moving flux tubes have magnetic field strengths typically higher than the surrounding plasma. If the total (plasma plus magnetic) pressure inside the flux tube is equal to that of the ambient plasma outside, show that the small change in field strength δB (inside minus outside field) is related to a corresponding change in plasma pressure δp as follows:

$$\delta p/p_o = -2(\delta B/B_o)(1/\beta_o) \tag{8}$$

where the subscript 'o' indicates quantities outside the flux tube, and β, as usual, equals the ratio of plasma pressure to magnetic pressure.

Using this formula, calculate $\delta p/p_o$ for values: (i) $B_o = 1700\,\mathrm{nT}$, $\delta B = 10\,\mathrm{nT}$, $\beta_o = 0.05$; and (ii) $B_o = 1700\,\mathrm{nT}$, $\delta B = 25\,\mathrm{nT}$, $\beta_o = 0.05$.

Solution The sum of the magnetic and plasma pressures outside the flux tube may be written as $B_o^2/(2\mu_o) + p_o$. If this quantity remains constant as we cross into the flux tube, we may express this by taking a zero differential between inside and outside, as follows: $d(B^2/(2\mu_o) + p) = 0 \approx 2B_o\,\delta B/(2\mu_o) + \delta p$.

Rearranging and dividing by p_o, we obtain $\delta p/p_o \approx -B_o\,(\delta B/\mu_o)(1/p_o) = -2(\delta B/B_o)(1/\beta_o)$, since, by definition $p_o = \beta_o(B_o^2/(2\mu_o))$.

Using this approximation and the values given, we obtain values of $\delta p/p_o$ of about (i) -0.24 and (ii) -0.59.

6. Consider the typical information for Mercury and the Earth in the table from the lecture notes which compares the magnetopause stand-off distances of various planets. Assuming that the dipole magnetic pressure of the planet balances solar wind dynamic pressure at the magnetopause standoff point, calculate the ratio of solar wind dynamic pressures just upstream of Mercury's and the Earth's magnetospheres.

   Solution

   The table in question indicates that the dipole magnetic pressure at Mercury's dayside magnetopause is approximately proportional to (ignoring dipole tilt effects) $[M_M/(1.4R_M)^3]^2$ (i.e. the magnetic pressure is proportional to the square of the expected field strength). Here M_M is Mercury's magnetic dipole moment. For the Earth, this quantity will be $[M_E/(10R_E)^3]^2$. Taking the ratio, we obtain $(M_M/M_E)^2\,(10^6/1.4^6)(R_E/R_M)^6$. Using reasonable values of the planetary radii, this evaluates to ~ 6.7. (N.B. I think the value of the magnetic moment of Mercury should be more like $4 \times 10^{-4}\,M_E$, based on Messenger data - note also the usual variability expected in solar wind parameters).

7. Chapman and Ferraro (1930) developed a model of a plasma cloud interacting with the Earth's dipole magnetic field. This model may be applied to investigate the behaviour of the magnetic field generated by the magnetopause currents. In this picture, the Earth's magnetic dipole is situated at the origin (Earth centre) and the dipole axis is orthogonal to the upstream solar wind direction. The magnetopause is then modelled as an infinite conducting plane, perpendicular to the upstream solar wind velocity, and situated a perpendicular distance of R_{MP} from the planet's dipole axis. Magnetopause currents flow on this plane and generate an *additional* field within the Earth's magnetosphere which is *equivalent* to that of an identical magnetic dipole, known as the *'image dipole'*, situated outside the magnetosphere at a distance $2R_{MP}$ from the Earth's centre along the direction anti-parallel to the upstream solar wind velocity. We define the x axis to pass through the Earth's centre (where x=0) along this direction.

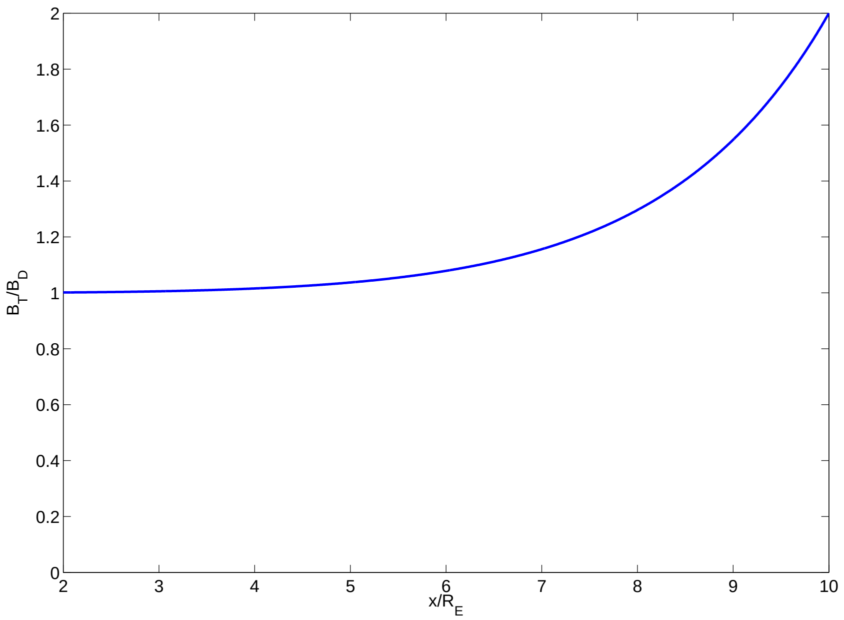
   Using this model, calculate and make a plot of the ratio B_{TOT}/B_{DIP} as a function of distance along the x axis, from the Earth's surface to the magnetopause plane. Here, B_{TOT} is the total magnetic field strength due to the actual and image dipoles combined, and B_{DIP} is the field strength due to the planetary dipole alone.

   Solution

   For the planetary dipole alone, the field strength outside the Earth and inside the magnetopause, along the x axis, is given by the function $B_D(x) = (B_E R_E^3/|x|^3)$ (using the nomenclature of Question 4). Now we may express the field of the image dipole situated at $x = 2R_{MP}$ as the function $B_D(x - 2R_{MP}) = (B_E R_E^3/|x - 2R_{MP}|^3)$. Adding the two, we obtain:

$$B_T(x) = B_D(x)\,(1 + |x|^3/|x - 2R_{MP}|^3).$$

   Hence $B_T(x)/B_D(x) = (1 + |x|^3/|x - 2R_{MP}|^3)$, which is always greater than unity. A plot of this quantity versus x/R_E is given below, using a reasonable value $R_{MP} = 10\,R_E$.



B_T/B_D
x/R_E
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
2
3
4
5
6
7
8
9
10