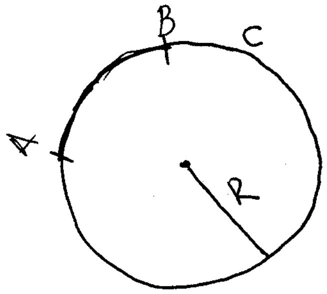
Review - trigonometry

1. Properties of circles

(*) Consider a circle delimited by the line C. All points on C are equally distant from the center of the circle.



1.1 Definitions:

- C = "circumference" (line delimiting a circle)
- R = "radius" (distance from the center to a point on the circumference)
- d = 2R = diameter
- "piece" AB on the circumference : arc

(*) Relation between the circumference and the diameter: $\frac{C}{d} = \pi$

($\pi = 3.1415927\ldots$)

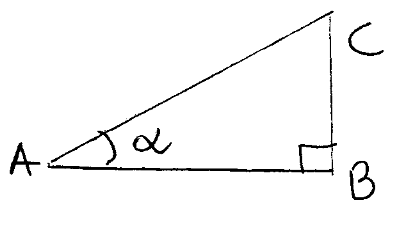
1.2 - Circular measure (radians): an angle α, measured in radians, delimits an arc of length $R\alpha$

(*) Conversion factors:

- degrees - radians : 1 radian — $\frac{180}{\pi}$ degrees
- radians - degrees : 1 degree — $\frac{\pi}{180}$ radians

2. Trigonometric functions

Consider the right-angled triangle ABC and the angle α



Then

(a) $\sin\alpha = \frac{BC}{AC} \Rightarrow \alpha = \mathrm{Arcsin}\left(\frac{BC}{AC}\right) = \sin^{-1}\left(\frac{BC}{AC}\right)$

(b) $\cos\alpha = \frac{AB}{AC} \Rightarrow \alpha = \mathrm{Arc\,cos}\left(\frac{AB}{AC}\right) = \cos^{-1}\left(\frac{AB}{AC}\right)$

(c) $\tan\alpha = \frac{BC}{AB} \Rightarrow \alpha = \mathrm{Arctan}\left(\frac{BC}{AB}\right)$
$\tan^{-1}\left(\frac{BC}{AB}\right)$

(*) Please note: i) $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ (obtained dividing (a) by (b))

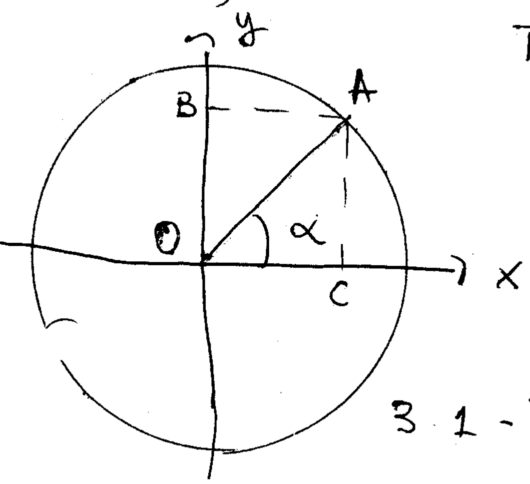
ii) $\sin^2\alpha + \cos^2\alpha = 1$

Proof: • from Pythagoras's theorem: $(AC)^2 = (AB)^2 + (CB)^2$

• dividing by $(AC)^2$: $1 = \underbrace{\left(\frac{AB}{AC}\right)^2}_{\cos^2\alpha} + \underbrace{\left(\frac{CB}{AC}\right)^2}_{\sin^2\alpha}$

3. Trigonometric circle:

Let us consider a circle of radius 1 in the xy plane, centered at the origin O, and let us define an angle α with respect to the x axis (by convention, α increases in the anti-clockwise direction)



Then:

• $\sin\alpha = \frac{\overline{OB}}{1}$ (projection onto the y axis)

• $\cos\alpha = \frac{\overline{OC}}{1}$ (projection onto the x axis)

3.1 - Particular cases

(a) $\alpha = 0 \Rightarrow \cos\alpha = 1$
$\sin\alpha = 0$

(b) $\alpha = \frac{\pi}{2} \Rightarrow \cos\alpha = 0$
$\sin\alpha = 1$

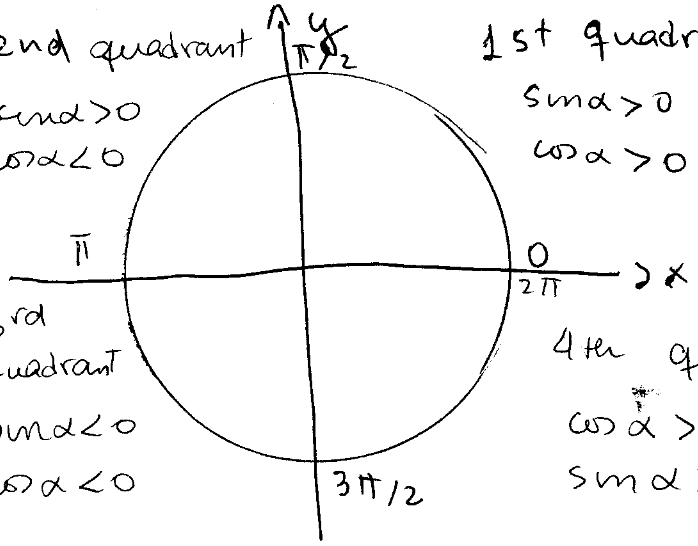
(c) $\alpha = \pi \Rightarrow \cos\alpha = -1$
$\sin\alpha = 0$

(d) $\alpha = \frac{3\pi}{2} \Rightarrow \cos\alpha = 0$
$\sin\alpha = -1$

3.2 - Signs of $\sin\alpha$, $\cos\alpha$

2nd quadrant
$\sin\alpha > 0$
$\cos\alpha < 0$

1st quadrant
$\sin\alpha > 0$
$\cos\alpha > 0$



3rd quadrant
$\sin\alpha < 0$
$\cos\alpha < 0$

4th quadrant
$\cos\alpha > 0$
$\sin\alpha > 0$

$$\sin(\alpha + 2\pi) = \sin\alpha$$
$$\cos(\alpha + 2\pi) = \cos\alpha$$

$\Rightarrow$ $\sin\alpha$, $\cos\alpha$ are periodic in 2π

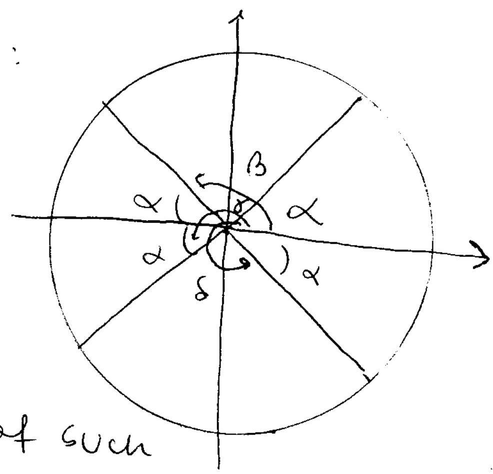
3.3 - Relations between cos, sin of particular angles

Let us consider the following angles:

(a) $\beta = \pi - \alpha$

(b) $\gamma = \pi + \alpha$

(c) $\delta = 2\pi - \alpha$, or $-\alpha$



• We wish to relate the sin and the cos of such angles to $\cos\alpha$, $\sin\alpha$

(a) $\sin\beta = \sin\alpha$, $\cos\beta = -\cos\alpha$ $\Rightarrow$ $\sin(\pi - \alpha) = \sin\alpha$, $\cos(\pi - \alpha) = -\cos\alpha$

(b) $\sin\gamma = -\sin\alpha$, $\cos\gamma = -\cos\alpha$ $\Rightarrow$ $\sin(\pi + \alpha) = -\sin\alpha$, $\cos(\pi + \alpha) = -\cos\alpha$

(d) $\sin\delta = -\sin\alpha$, $\cos\delta = \cos\alpha$ $\Rightarrow$ $\sin(2\pi - \alpha) = -\sin\alpha$, $\cos(2\pi - \alpha) = \cos\alpha$ or $\sin(-\alpha) = -\sin\alpha$, $\cos(-\alpha) = \cos\alpha$

((d) shows that $\sin\alpha$ is an ODD function and $\cos\alpha$ is an EVEN function)

(*) Note that $-1 < \sin\alpha < 1$, $-1 < \cos\alpha < 1$ $\Rightarrow$ both functions are bounded

4. Other trigonometric functions

(a) $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ $\Rightarrow$ goes to infinity when $\cos\alpha = 0$, i.e., for $\alpha = (2n+1)\frac{\pi}{2}$

(b) $\text{cotan}\,\alpha = \frac{1}{\tan\alpha}$

(c) $\sec\alpha = \frac{1}{\cos\alpha}$

(d) $\text{cosec}\,\alpha = \frac{1}{\sin\alpha}$

(*) Please note: (b), (c) and (d) are <u>NOT</u> the inverse trigonometric functions $\tan^{-1}\alpha$, $\cos^{-1}\alpha$, $\sin^{-1}\alpha$

4.1. Some trigonometric identities

(a) $\tan^2\alpha + 1 = \sec^2\alpha$

Proof: $\frac{\sin^2\alpha}{\cos^2\alpha} + 1 = \frac{\sin^2\alpha + \cos^2\alpha}{\cos^2\alpha} = \frac{1}{\cos^2\alpha}$

(b) $\text{cotan}^2\alpha + 1 = \text{cosec}^2\alpha$

Proof: $\frac{\cos^2\alpha}{\sin^2\alpha} + 1 = \frac{\cos^2\alpha + \sin^2\alpha}{\sin^2\alpha} = \frac{1}{\sin^2\alpha}$

$\rightarrow$ see back

5. <u>Calculus with trigonometric functions</u>

6.1 - <u>Differentiating</u>

(a) $\frac{d}{d\alpha}\sin\alpha = \cos\alpha$

(b) $\frac{d}{d\alpha}(\cos\alpha) = -\sin\alpha$

(c) $\frac{d}{d\alpha}(\tan\alpha) = 1 + \tan^2\alpha = \sec^2\alpha$

(*) Proof of (c): $\frac{d}{d\alpha}(\tan\alpha) = \frac{d}{d\alpha}\left(\frac{\sin\alpha}{\cos\alpha}\right)$

Applying the quotient rule $\left(\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}\right)$, we obtain

(4a)

5. <u>Addition formulae</u>:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

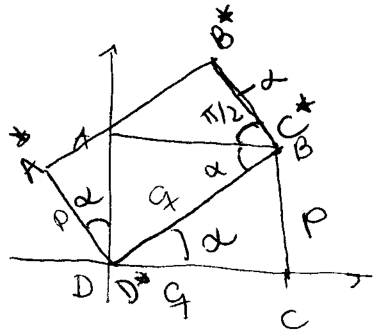
<u>Proof</u>: Let us consider the rectangle ABCD with the coordinates

$A = (0, p)$

$B = (q, p)$

$C = (q, 0)$

$D = (0, 0)$



we now rotate this rectangle in an angle α. We wish to determine its new coordinates

$A^* = (-p\sin\alpha, p\cos\alpha)$

$B^* = (q\cos\alpha - p\sin\alpha, q\sin\alpha + p\cos\alpha)$

$C^* = (q\cos\alpha, q\sin\alpha)$

$D^* = (0, 0)$

One may write the new coordinates as

$$A^* = \begin{pmatrix} 0 \\ p \end{pmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} ; \quad B^* = \begin{bmatrix} q \\ p \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$C^* = \begin{pmatrix} q \\ 0 \end{pmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} ; \quad D^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

(i.e. as a vector representing the old coordinates multiplied by a matrix)

Hence, a rotation through α radians is described by $M_{(\alpha)} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Let us now consider a rotation through $\alpha+\beta$ radians.

This can be described as a rotation by α followed by a rotation by β radians.

Hence $M_{(\alpha+\beta)} = M_{(\alpha)} M_{(\beta)}$

$$\Rightarrow \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}}$$

$$\begin{bmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\beta\cos\alpha - \sin\alpha\cos\beta \\ \sin\alpha\cos\beta + \sin\beta\cos\alpha & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \sin(\alpha+\beta) = \sin\beta\cos\alpha + \sin\alpha\cos\beta \end{array}$$

$$\frac{d}{d\alpha}(\tan\alpha) = \frac{\cos\alpha \frac{d}{d\alpha}\sin\alpha - \sin\alpha \frac{d}{d\alpha}\cos\alpha}{\cos^2\alpha} = \frac{\overbrace{\cos^2\alpha + \sin^2\alpha}^{1}}{\cos^2\alpha}$$

$$\Rightarrow \frac{d}{d\alpha}(\tan\alpha) = \frac{1}{\cos^2\alpha}$$

6.2 - Integrating

6.2.1 - Direct trigonometric functions

(a) $\int \cos x \, dx = \sin x + C$

(b) $\int \sin x \, dx = -\cos x + C$

(c) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$ (performed by substitution)

$u = \cos x$

$du = -\sin x \, dx$

6.2.2 - Integrals which yield inverse trigonometric functions

(at least those we have seen in class)

$\int \frac{1}{1+x^2} dx = \text{Arc}\tan x + C$

$\int \frac{1}{\sqrt{1-x^2}} dx = \text{Arc}\sin x + C$

can all be proven by substitution

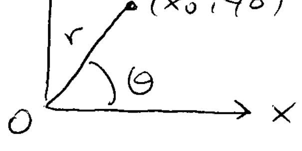
7. Polar coordinates

7.1 - Generalities

(*) In general, one uses coordinates to locate something (e.g, a point) in a plane/in space

Problem: Consider a point in a plane, whose distance from the origin of the coordinate system is r. How to locate this point?

Example 1: cartesian coordinates: we take the projections of r onto the x, y axis



Example 2: Polar coordinates: we take the distance r from the origin and the angle θ $\overline{OP}$ forms with the x axis

7.2 Relation between cartesian and polar coordinates:

$x = r\cos\theta$; $y = r\sin\theta$

On the other hand, using Pythagoras's theorem,

$r = [x^2+y^2]^{1/2}$

Furthermore,

$$\cos\theta = \frac{x}{r} \Rightarrow \theta = \text{Arc}\cos\left(\frac{x}{r}\right) \qquad (*)$$

or

$$\sin\theta = \frac{y}{r} \Rightarrow \theta = \text{Arc}\cos\left(\frac{y}{r}\right) \qquad (**)$$

or

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \text{Arc}\tan\left(\frac{y}{x}\right) \qquad (***)$$

- Normally one uses (***) to determine θ
- Please note: the above definitions not sufficent on their own to determine a unique angle

- <u>Example</u>: $\tan\left(\frac{\pi}{4}\right) = 1$

  and

  $\tan\left(\pi + \frac{\pi}{4}\right) = 1$

- way to find θ:

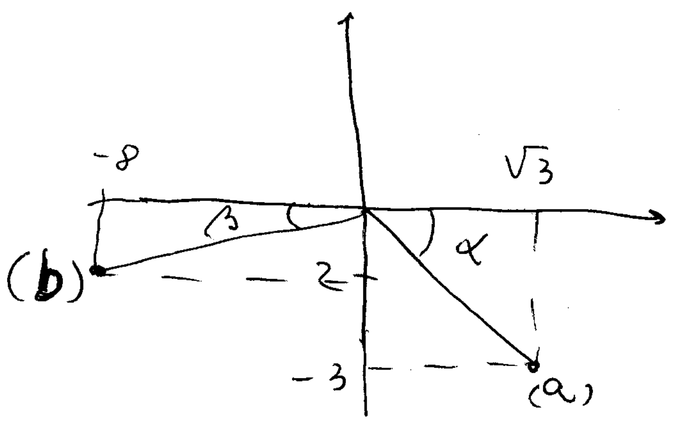
  (a) find an acute angle α so that $\tan\alpha = \left|\frac{y}{x}\right|$

  (b) θ is α, $-\alpha$, $\alpha - \pi$ or $\pi - \alpha$ depending on the signs of x and y

- <u>Example</u>: Find the polar coordinates of the following points:

  (a) $(x, y) = (\sqrt{3}, -3)$

  (b) $(x, y) = (-$[illegible]$, -2)$



(a) Find r:

$r = \sqrt{(x)^2 + (y)^2} = \sqrt{3+9}$

$= \sqrt{12} = 2\sqrt{3}$

Find θ:

$\alpha = \text{Arc}\tan\left|\frac{y}{x}\right| = \text{Arc}\tan\left|\frac{1}{\sqrt{3}}\right|$

$\alpha = \frac{\pi}{3}$

Since, however, y has a negative coordinate, $-\frac{\pi}{2} < \theta < 0$