1

Find P_{12}:
$$P_{012} = \frac{(x-x_2)P_{01} + (x_0-x)P_{12}}{(x_0-x_2)}$$

$$\frac{27}{\cancel{7}} = \frac{(0.5-0.7)\cdot 3.5 - 0.5P_{12}}{-0.\cancel{7}1}$$

$$27 = \frac{0.7 + 0.5P_{12}}{0.1}$$

$$7 + 5P_{12} = 27$$

$$\boxed{P_{12} = 4}$$

Find P_2:
$$P_{12} = \frac{(x-x_2)P_1 + (x_1-x)P_2}{(x_1-x_2)}$$

$$4 = \frac{(-0.2)\cdot 2.8 + \overbrace{(0.4-0.5)}^{-0.1}P_2}{-0.3}$$

$$4 = \frac{0.56 + 0.1P_2}{0.3}$$

$$5.6 + P_2 = 12 \Rightarrow \boxed{P_2 = 6.4}$$

$\nu = \mu = 3$

$f(x) = \sin x \Rightarrow$ Taylor expansion (up to 6th order):

$$f(x) \cong x - \frac{x^3}{6} + \frac{x^5}{120} + O[x^7]$$

$a_0 = 0$; $a_1 = 1$; $a_2 = 0$; $a_3 = -1/6$; $a_4 = 0$;
$a_5 = 1/120$

$$R_{33} = \frac{p_0 + p_1 x + p_2 x^2 + p_3 x^3}{1 + q_1 x + q_2 x^2 + q_3 x^3}$$

Recurrence formulae $\left[\text{from } \sum_{i=0}^{k} a_i q_{k-i} = p_k \quad k = 0, 1, \ldots, N \right]$

$a_0 = p_0 \Rightarrow p_0 = 0$

$a_1 q_0 + \underbrace{a_0}_{=0} q_1 = p_1 \Rightarrow p_1 = 1$

$a_0 q_2 + a_1 q_1 + a_2 q_0 = p_2 \Rightarrow q_1 = p_2$

$a_0 q_3 + a_1 q_2 + a_2 q_1 + a_3 q_0 = p_3 \Rightarrow q_2 - \frac{1}{6} = p_3$

$a_1 q_3 + a_2 q_2 + a_3 q_1 + a_4 q_0 = 0 \Rightarrow q_3 - \frac{1}{6} q_1 = 0$

$a_2 q_3 + a_3 q_2 + a_5 q_0 = 0 \Rightarrow -\frac{1}{6} q_2 + \frac{1}{120} = 0$

$a_3 q_3 = 0 \Rightarrow q_3 = 0$

$q_3 = 0$; $q_2 = \frac{1}{20}$; $q_1 = 0$; $p_3 = -\frac{7}{60}$; $p_2 = 0$; $p_1 = 1$; $p_0 = 0$

$$\boxed{R_{33} = \frac{x - \frac{7}{60} x^3}{1 + \frac{1}{20} x^2}}$$

Comparison $f(x)$, $R_{33}(x)$ and $f_6(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$

x_i	$\frac{\lvert f(x) - R_{33}(x)\rvert}{\lvert f(x)\rvert}$	$\frac{\lvert f(x) - f_6(x)\rvert}{\lvert f_6(x)\rvert}$
0.2	1.40×10^{-8}	1.28×10^{-8}
0.4	9.06×10^{-7}	8.33×10^{-7}
0.6	1.05×10^{-5}	9.79×10^{-6}

(*) The relative errors obtained with the Padé approx. $R_{33}(x)$ are comparable to those obtained using the sixth-order Taylor expansion of $f(x)$

3

$f(x) = \cos \pi x$

x	$f(x)$	
0	1	a_0
0.25	0.707107 $(1/\sqrt{2})$	a_1
0.5	0	a_2
0.75	−0.707107 $(-1/\sqrt{2})$	a_3
1.0	−1	a_4

$h_0 = h_1 = \cdots = h_4 = 0.25$

Tridiagonal system to be solved: $A\vec{x} = \vec{b}$ with

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0.25 & 0 & 0 \\ 0 & 0.25 & 1 & 0.25 & 0 \\ 0 & 0 & 0.25 & 1 & 0.25 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$\vec{b} = \begin{bmatrix} 0 \\ 12(1-\sqrt{2}) \\ 0 \\ 12(-1+\sqrt{2}) \\ 0 \end{bmatrix}$$

$c_0 = 0$

$c_4 = 0$

$0.25c_2 + c_1 = 12(1-\sqrt{2})$

$c_2 + 0.25c_1 + 0.25c_3 = 0$

$0.25c_2 + c_3 = 12(-1+\sqrt{2})$

$\Rightarrow c_2 = 0$

$c_3 = 12(\sqrt{2}-1) = 4.97056$

$c_1 = -4.97056$

$b_0 = \frac{1}{h}(a_1 - a_0) - \frac{h}{3}(c_1 + 2c_0) = -0.757359$

$b_1 = \frac{1}{h}(a_2 - a_1) - \frac{h}{3}(c_2 + 2c_1) = -2$

$b_2 = \frac{1}{h}(a_3 - a_2) - \frac{h}{3}(c_3 + 2c_2) = -3.2426$

$b_3 = \frac{1}{h}(a_4 - a_3) - \frac{h}{3}(c_4 + 2c_3) = -1.89$

$d_0 = \frac{4}{3}c_1 = -6.62742$

$d_1 = \frac{4}{3}(c_2 - c_1) = +6.62742$

$d_2 = \frac{4}{3}(c_3 - c_2) = 6.62742$

$d_4 = -6.62742$

Splines:

$S_0(x) = a_0 + b_0 x + c_0 x^2 + d_0 x^3 \quad , 0 \le x < 0.25$

$S_1(x) = a_1 + b_1(x-0.25) + c_1(x-0.25)^2 + d_1(x-0.25)^3 , \; 0.25 \le x < 0.$

$S_2(x) = a_2 + b_2(x-0.5) + c_2(x-0.5)^2 + d_2(x-0.5)^3 , \; 0.5 \le x \le 0.7$

$S_3(x) = a_3 + b_3(x-0.75) + c_3(x-0.75)^2 + d_3(x-0.75)^3 , \; 0.75 \le x < 1$

$$\int_0^{0.5} f(x)dx \approx \int_0^{0.25} S_0(x)\,dx + \int_{0.25}^{0.5} S_1(x)\,dx = 0.314721$$

For comparison: $\frac{1}{\pi} \approx 0.31831$

$f'(0.3) \qquad S_1'(x)\Big|_{x=0.3} = b_1 + 2c_1(x-0.25) + 3d_1(x-0.25)^2 = -2.447$

for comparison: $-\pi \sin(\pi x)\Big|_{x=0.3} = -2.5416$

$f'(0.5)$ → One can take either $S_1'(x)$ or $S_2'(x)$, because we imposed that the 1st and 2nd derivatives of the splines are continuous

Result using $S_2'(x) = -3.24264$

$S_1'(x) = -3.24264$

For comparison: $-\pi \sin(\pi x)\Big|_{x=0.5} = -3.14159$

4

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & 1 \le x \le 2 \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^2, & 2 \le x \le \end{cases}$$

$f'(1) = f'(3)$

$f'(3)$

From $S_0(x)$:

$a_0 = 0$

$b_0 = 3$ $\quad f'(3) = b_1 = S_1'(3)$

$c_0 = 2$ $\quad f'(1) = b_0 = S_0'(1)$

$d_0 = -1$ $\quad \boxed{b_1 = 3}$

$h_0(2c_0 + c_1) = -3f'(1) + \frac{3}{h}(a_1 - a_0) \Rightarrow 1(4+c) = -9 + 3a$

$b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n)$ $\qquad 2 = -9 + 3a$

$3 = 3 + (2 + c) \Rightarrow 2 + c = 0 \Rightarrow \boxed{c = -2}$ $\qquad 3a = 11$

$\boxed{d = \frac{c - c_0}{3} = \frac{-4}{3}}$ $\qquad \boxed{a = \frac{11}{3}}$

(5) $u(x) = f(x) + \int_a^b k(x,t)\,u(t)\,dt$

$u(x_i) = f(x_i) + \int_a^b k(x_i,t)\,u(t)\,dt$

$x_i \rightarrow$ nodes upon which integrating methods are based

Trapezoidal rule: $\int_a^b f(x)\,dx \simeq \frac{1}{2}(f(b)+f(a))(b-a)$

Nodes: a, b

$$u(a) = f(a) + \underbrace{\int_a^b k(a,t)\,u(t)\,dt}_{\approx \frac{1}{2}[k(a,a)\,u(a) + k(a,b)\,u(b)](b-a)}$$

$$u(b) = f(b) + \underbrace{\int_a^b k(b,t)\,u(t)\,dt}_{\approx \frac{1}{2}[k(b,a)\,u(a) + k(b,b)\,u(b)](b-a)}$$

(*) Specific case : $a=0;\ b=1$

$u(0) = f(0) + \frac{1}{2}[k(0,0)\,u(0) + k(0,1)\,u(1)]$

$u(1) = f(1) + \frac{1}{2}[k(1,0)\,u(0) + k(1,1)\,u(1)]$

$f(x) = x^2$; $k(x,t) = \exp[|x-t|]$

$$\underbrace{\begin{aligned} u(0) &= \tfrac{1}{2}u(0) + \tfrac{1}{2}e\,u(1) &\Rightarrow& -\tfrac{1}{2}u(0) + \tfrac{e}{2}u(1) = 0 \\ u(1) &= 1 + \tfrac{e}{2}u(0) + \tfrac{1}{2}u(1) &\Rightarrow& \tfrac{e}{2}u(0) - \tfrac{1}{2}u(1) = -1 \end{aligned}}_{\text{linear syst. of equations}}$$

(b) Composite trapezoidal rule

$(n=4)$

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right]$$

$$n=4 \Rightarrow \int_a^b f(x)\,dx = \frac{h}{2}\Big[\underset{\substack{\uparrow\\ x_0}}{f(a)} + 2\sum_{j=1}^{3} f(x_j) + \underset{\substack{\uparrow\\ x_4}}{f(b)}\Big]$$

$$h = \frac{b-a}{n} = \frac{1}{4}$$

Nodes : x_0, x_1, x_2, x_3, x_4

$$u(x_0) = f(x_0) + \int_a^b k(x_0,t)\,u(t)\,dt$$

$$u(x_1) = f(x_1) + \int_a^b k(x_1,t)\,u(t)\,dt$$

$$u(x_2) = f(x_1) + \int_a^b k(x_2,t)\,u(t)\,dt$$

$$u(x_3) = f(x_3) + \int_a^b k(x_3,t)\,u(t)\,dt$$

$$u(x_4) = f(x_4) + \int_a^b k(x_4,t)\,u(t)\,dt$$

Linear syst. of equations

$$u(x_j) = f(x_j) + \frac{h}{2}\Big[k(x_j,x_0)\,u(x_0) + k(x_j,x_0+h)\,u(x_0+h)$$
$$+ k(x_j,x_0+2h)\,u(x_0+2h) +$$
$$+ k(x_j,x_0+3h)\,u(x_0+3h) +$$
$$+ k(x_j,x_0+4h)\,u(x_0+4h)\Big]$$

$(j = 0,1,\dots,4)$

Specifically,

$$u(0) = \underbrace{f(0)}_{\overset{\shortparallel}{0}} + \frac{1}{8}\Big[\underbrace{k(0,0)}_{1} \cdot U(0) + \overbrace{k(0,1/4)}^{e^{1/4}}u(1/4) + \overbrace{k(0,1/2)}^{e^{1/2}}u(1/2)$$

$$+ \underbrace{k(0,3/4)}_{e^{3/4}}u(3/4) + e\, u(1)\Big]$$

$$U(1/4) = \frac{1}{16} + \frac{1}{8}\Big[e^{1/4}u(0) + 1 \cdot U(1/4) + e^{1/4}U(1/2) + e^{1/2}U(3/4) +$$

$$+ e^{3/4}u(1)\Big]$$

$$U(1/2) = \frac{1}{4} + \frac{1}{8}\Big[e^{1/2}u(0) + e^{1/4}u(1/4) + U(1/2) + e^{1/4}U(3/4)$$

$$+ e^{1/2}U(1)\Big]$$

$$U(3/4) = \frac{9}{16} + \frac{1}{8}\Big[e^{3/4}U(0) + e^{1/2}U(1/4) + e^{1/4}U(1/2) + U(3/4) + e^{1/4}U(1)\Big]$$

$$U(1) = 1 + \frac{1}{8}\Big[e\,u(0) + e^{3/4}U(1/4) + e^{1/2}U(1/2) + e^{1/4}U(3/4) + U(1)\Big]$$

Linear system of equations :

E_1 : $-7u(0) + e^{1/4}U(1/4) + e^{1/2}U(1/2) + e^{3/4}U(3/4) + e\,U(1) = 0$

E_2 : $e^{1/4}U(0) - 7U(1/4) + e^{1/4}U(1/2) + e^{1/2}U(3/4) + e^{3/4}U(1) = -\frac{1}{2}$

E_3 : $e^{1/2}U(0) + e^{1/4}U(1/4) - 7U(1/2) + e^{1/4}U(3/4) + e^{1/2}U(1) = -2$

E_4 : $e^{3/4}U(0) + e^{1/2}U(1/4) + e^{1/4}U(1/2) - 7U(3/4) + e^{1/4}U(1) = -\frac{9}{2}$

E_5 : $e\,u(0) + e^{3/4}u(1/4) + e^{1/2}U(1/2) + e^{1/4}U(3/4) - 7U(1) = -8$

Augmented matrix of the syst. :

$$A = \left[\begin{array}{ccccc:c}
-7 & e^{1/4} & e^{1/2} & e^{3/4} & e & 0 \\
e^{1/4} & -7 & e^{1/4} & e^{1/2} & e^{3/4} & -1/2 \\
e^{1/2} & e^{1/4} & -7 & e^{1/4} & e^{1/2} & -2 \\
e^{3/4} & e^{1/2} & e^{1/4} & -7 & e^{1/4} & -9/2 \\
e & e^{3/4} & e^{1/2} & e^{1/4} & -7 & -8
\end{array}\right]$$

Solution (checked & performed with mathematica)

$U(1) = 31.862$

$U(3/4) = 26.61$

$U(1/2) = 24.8$

$U(1/4) = 26.2236$

$U(0) = 31.0719$

(looks pretty messy; one should check for round-off errors)