Exercise sheet 5

1

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

Cholesky decomposition: $A = L \underbrace{L^T}_{U}$

$l_{11} = \sqrt{2}$

- 1^{st} column of l / 1^{st} row of U

$l_{21} = u_{12} = -\frac{1}{\sqrt{2}}$

$l_{31} = u_{13} = 0$

- 2^{nd} element of main diagonal

$$l_{22} = (a_{22} - l_{21}^2)^{1/2} = \left(2 - \frac{1}{2}\right)^{1/2} = \left(\frac{3}{2}\right)^{1/2}$$

- $$l_{32} = \frac{1}{l_{22}}\left[a_{32} - l_{31} l_{21}\right] = \left(\frac{2}{3}\right)^{1/2} \times -1 = u_{23} = -\left(\frac{2}{3}\right)^{1/2}$$

$$l_{33} = (2 - l_{31}^2 - l_{32}^2)^{1/2} = \left(2 - \frac{2}{3}\right)^{1/2} = \frac{2}{\sqrt{3}}$$

$$L = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ -1/\sqrt{2} & \sqrt{3/2} & 0 \\ 0 & -\sqrt{2/3} & \frac{2}{\sqrt{3}} \end{bmatrix} \qquad U = L^T = \begin{bmatrix} \sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & \sqrt{3/2} & -\sqrt{2/3} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

2 Tridiagonal system:

$$\begin{aligned} 2x_1 + x_2 \quad\quad &= 3 \\ x_1 + 2x_2 + x_3 &= -2 \\ 2x_2 + 3x_3 &= 0 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix}}_{} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}}_{\vec{b}}$$

A (matrix of the system)

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 3 \end{bmatrix}$$

CROUT factorization algorithm

$l_{11} = a_{11} \Rightarrow l_{11} = 2$

$u_{12} = a_{12}/l_{11} = 1/2$

For $i = 2, \ldots, \overbrace{n-1}^{2}$

$l_{i,i-1} = a_{i,i-1}$

$l_{ii} = a_{ii} - l_{i,i-1}\, u_{i-1,i}$

$u_{i,i+1} = a_{i,i+1}/l_{ii}$

$\Rightarrow$

$l_{21} = a_{21} = 1$

$l_{22} = a_{22} - l_{21} u_{12} = 2 - \frac{1}{2} = \frac{3}{2}$

$u_{23} = \frac{3}{3/2} = 2$

$l_{32} = a_{32} = 2$

$l_{33} = a_{33} - l_{32} u_{23} = 3 - 2 \cdot 2 = -1$

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3/2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3/2 & 0 \\ 0 & 2 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

• Solve for y:

$$2y_1 = 3 \Rightarrow y_1 = 3/2$$

$$1 \cdot y_1 + 3/2\, y_2 = -2 \Rightarrow \frac{3}{2} y_2 = -2 - \frac{3}{2} = -\frac{7}{2}$$

$$y_2 = -\frac{7}{3}$$

$$2y_2 - y_3 = 0 \Rightarrow y_3 = 2y_2 = \frac{-14}{3}$$

• Solve for x:

$$\begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3/2 \\ -7/3 \\ -14/3 \end{bmatrix}$$

$$x_3 = -\frac{14}{3}$$

$$x_2 + 2x_3 = -7/3$$

$$x_2 = \frac{-7}{3} + \frac{28}{3} = \frac{21}{3} = 7$$

$$x_1 + \frac{x_2}{2} = 3/2$$

$$x_1 + \frac{7}{2} = 3/2 \qquad x_1 = -\frac{4}{2} = -2$$

$$\boxed{\vec{x} = \begin{bmatrix} -2 \\ 7 \\ -14/3 \end{bmatrix}}$$

3 Lagrange polynomial

$$P_3(x) = \underbrace{f(x_0)}^{0.1924} L_0(x) + \overbrace{f(x_1)}^{0.2414} L_1(x) + \overbrace{f(x_2)}^{0.2933} L_2(x) + \overbrace{f(x_3)}^{0.3492} L_3(x)$$

with

$$L_0(x) = \frac{(x-1.05)(x-1.10)(x-1.15)}{\underbrace{(-0.05)(-0.1)(-0.15)}_{-7.5\times10^{-4}}}$$

$$L_1(x) = \frac{(x-1.00)(x-1.10)(x-1.15)}{\underbrace{(0.05)(-0.05)(-0.1)}_{2.5\times10^{-4}}}$$

$$L_2(x) = \frac{(x-1.00)(x-1.05)(x-1.15)}{\underbrace{(0.1)(0.05)(-0.05)}_{-2.5\times10^{-4}}}$$

$$L_3(x) = \frac{(x-1.00)(x-1.05)(x-1.10)}{\underbrace{(0.15)(0.1)(0.05)}_{7.5\times10^{-4}}}$$

$f_2[1.09] = 0.2826$ (four digit rounding arithmetic)