1. $A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ -1 & 3 & 3 & 0 \\ -2 & -2 & 1 & 4 \\ -2 & 2 & 2 & 5 \end{bmatrix} \begin{matrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{matrix}$

Given : main elements a_{ij}

main diagonal of L: $l_{11} = l_{22} = l_{33} = l_{44} = 1$

- 1st row of U:

$u_{11} = a_{11} = 2$

$u_{12} = a_{12} = 1$

$u_{13} = a_{13} = 0$

$u_{14} = a_{14} = 0$

- 1st column of L: $l_{21} = a_{21}/a_{11} = -1/2$

$l_{31} = a_{31}/a_{11} = -1$

$l_{41} = a_{41}/a_{11} = -1$

- 2nd row of U:

$u_{21} = 0$

$u_{22} = a_{22} - l_{21} u_{12} = 3 + \frac{1}{2} \cdot 1 = 7/2$

$u_{23} = a_{23} - l_{31} u_{13} = 3$

$u_{24} = a_{24} - l_{41} u_{14} = 0$

(*) Please note: this is what you get if you perform the Gaussian elimination

$(E_2 + \frac{1}{2} E_1) \to (E_2)$ $\quad l_{21} = -1/2$

$(E_3 + E_1) \to (E_3)$ $\quad l_{31} = -1$

$(E_4 - E_1) \to (E_4)$ $\quad l_{41} = -1$

(the intermediate matrix is

$$A^{(2)} = \begin{array}{c} \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 7/2 & 3 & 0 \\ 0 & -1 & 1 & 4 \\ 0 & 3 & 2 & 5 \end{bmatrix} \end{array} \begin{array}{l} E_1 \\ E_2^{(2)} \\ E_3^{(2)} \\ E_4^{(2)} \end{array}$$

, we modified other matrix elements apart from those already written have been modified; for that reason there are these sums in the algorithm when computing the rows of U/columns of L).

• 2^{nd} column of L:

$l_{22} = 1$

$$l_{32} = a_{32}^{(2)} / \underbrace{a_{22}^{(2)}}_{u_{22}} = -2/7 = \underbrace{\left[\overset{-2}{a_{32}} - \overset{-1}{l_{31}} \times \overset{1}{u_{12}}\right]}_{-2+1} / u_{22}$$

$$l_{42} = a_{42}^{(2)} / a_{22}^{(2)} = 3 \times \frac{2}{7} = \frac{6}{7}$$

• 3Rd row of U:

$$u_{31} = 0$$
$$u_{32} = 0$$
$$u_{33} = a_{33}^{(2)} - l_{32} u_{23} = a_{33} - l_{31} u_{13} - l_{32} u_{23} =$$
$$= 1 + \frac{2}{7} \times 3 = \frac{13}{7}$$
$$u_{34} = a_{34}^{(2)} - l_{32} u_{24} = 4$$

(we performed $\left(E_3^{(2)} + \frac{2}{7} E_2^{(2)}\right) \to \left(E_3^{(2)}\right)$

$\left(E_4^{(2)} + \frac{6}{7} E_2^{(2)}\right) \to \left(E_4^{(2)}\right)$)

• [illegible]

The intermediate matrix is

$$A^{(3)} = \begin{array}{c} \left[\begin{array}{cccc} 2 & 1 & 0 & 0 \\ 0 & 7/2 & 3 & 0 \\ 0 & 0 & 13/7 & 4 \\ 0 & 0 & -4/7 & 5 \end{array}\right] \end{array} \begin{array}{l} E_1 \\ E_2^{(2)} \\ E_3^{(3)} \\ E_4^{(3)} \end{array}$$

3^{Rd} column of L | 4^{th} row of U

$$\left(E_4^{(3)} + \frac{4}{7} \Big/ \frac{13}{7} \times E_3^{(3)}\right) \to \left(E_4^{(3)}\right)$$

$$\left(E_4^{(3)} + \frac{4}{13} E_3^{(3)}\right) \to \left(E_4^{(3)}\right) \qquad l_{43} = -4/13$$

$$u_{44} = \; + \frac{4}{13} \times 4 = 6.2308 = 81/13$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1/2 & 1 & 0 & 0 \\ -1 & -2/7 & 1 & \\ -1 & 6/7 & -4/13 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 7/2 & 3 & 0 \\ 0 & 0 & 13/7 & 4 \\ 0 & 0 & 0 & \frac{81}{13} \end{pmatrix}$$

2 To check this, we will perform Gaussian elimination in the matrices below

(a)
$$\begin{array}{cc} A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -1 \end{bmatrix} & \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \end{array}$$

$(E_2 - 2E_1) \rightarrow (E_2)$
$$A^{(2)} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$
Rows 2 and three must be interchange

Permutation matrix:
$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
interchanging rows
$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$
Problem: $a_{11} = 0$
Rows 1 and 2 must be interchanged

Are there further problems? Let us check

$$\begin{array}{cc} \tilde{A} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} & \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \end{array}$$
$(E_3 - E_1) \rightarrow (E_3)$

$$\tilde{A} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

NO probls!

3. $$\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}\underbrace{\begin{bmatrix}2&3&-1\\0&-2&1\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}} = \begin{bmatrix}-1\\3\\0\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\2&1&0\\-1&0&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} = \begin{bmatrix}-1\\3\\0\end{bmatrix}$$

- Forward substitution:

$$y_1 = -1$$
$$y_2 = 3 - 2y_1 = 5 \qquad \Rightarrow y = \begin{bmatrix}-1\\5\\-1\end{bmatrix}$$
$$y_3 = 0 + y_1 + 0 \cdot y_2 = y_1 = -1$$

$$\begin{bmatrix}2&3&-1\\0&-2&1\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-1\\5\\-1\end{bmatrix}$$

- Backward substitution

$$x_3 = -1/3$$
$$x_2 = \frac{-1}{2}\left[5 + \frac{1}{3}\right] = -\frac{8}{3}.$$
$$x_1 = \frac{1}{2}\left[-1 - 3x_2 + x_3\right] = \frac{1}{2}\left[-1 - 8 - \frac{1}{3}\right] = -\frac{14}{3}$$

$$\vec{x} = \begin{bmatrix} -14/3 \\ -8/3 \\ -1/3 \end{bmatrix}$$

4 Matrix of the system:

$$\begin{array}{cc} A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix} & \begin{array}{c} E_1 \\ E_2 \\ E_3 \end{array} \end{array}$$

LU Decomposition:

$l_{11} = l_{22} = l_{33} = 1$

$u_{11} = 2;\ u_{12} = -1;\ u_{13} = 3$

$(E_2 - \frac{3}{2} E_1) \rightarrow (E_2) \qquad l_{21} = 3/2$

$(E_3 - \frac{3}{2} E_1) \rightarrow (E_3) \qquad l_{31} = 3/2$

$$A^{(2)} = \begin{array}{c} E_1 \\ E_2^{(2)} \\ E_3^{(2)} \end{array} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 9/2 & 9/2 \\ 0 & 9/2 & 1/2 \end{bmatrix} \begin{array}{l} \leftarrow \text{1st row of } U \\ \leftarrow \text{2nd row of } U \\ \end{array}$$

$(E_3^{(2)} - E_2^{(2)}) \rightarrow (E_3^{(2)}) \qquad l_{32} = 1$

$$A^{(3)} = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 9/2 & 9/2 \\ 0 & 0 & -4 \end{bmatrix} = U \begin{array}{l} \\ \\ \leftarrow \text{third row of } U \end{array}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 9/2 & 9/2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & -1 & 3 \\ 0 & 9/2 & 9/2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

Solve for y:

$$\begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ 3/2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix} \rightarrow y_1 = -1$$

$$y_2 = -\frac{3}{2} \cdot y_1 = 3/2$$

$$y_3 = -\frac{3}{2} y_1 - y_2$$

$$= \frac{3}{2} - \frac{3}{2} = 0$$

Solve for x:

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 9/2 & 9/2 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3/2 \\ 0 \end{bmatrix}$$

$$x_3 = 0 \quad ; \quad 9/2\, x_2 = 3/2 \Rightarrow x_2 = \frac{1}{3}$$

$$2x_1 = -1 + x_2 = \frac{-2}{3} \Rightarrow x_1 = -\frac{1}{3}$$

$$\vec{x} = \begin{bmatrix} -1/3 \\ 1/3 \\ 0 \end{bmatrix}$$