Sheet 9 - solutions

1) $F(x,y) = 0$

(a) $\frac{dy}{dx} = ?$

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y}$$

$$\frac{\partial F}{\partial x} = 3x^2 - 4x + 3 \quad ; \quad \frac{\partial F}{\partial y} = -2x^2 + 6xy$$

$$\frac{dy}{dx} = \frac{-3x^2 + 4x - 3}{-2x^2 + 6xy}$$

(b) $\frac{\partial F}{\partial x} = 4x + 4y$

$\frac{\partial F}{\partial y} = 4x - 4y^3$ $\Rightarrow$ $\frac{dy}{dx} = \frac{-(4x+4y)}{4x - 4y^3}$

$$\frac{dy}{dx} = \frac{-(x+y)}{x - y^3}$$

(2) $f(x,y,z) = x y^{\alpha} z$

Sheet 9 $g(x,y,z) = x^{\beta} e^{yz}$

(*) Differentiating: $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$

$$dg = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz$$

Particular case: g, f const: $df = dg = 0$

$$\Rightarrow \quad \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$$

$$\frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy + \frac{\partial g}{\partial z} dz = 0$$

This yields

$$\frac{\partial f}{\partial x}\frac{dx}{dy} + \frac{\partial f}{\partial y}\overbrace{\frac{dy}{dy}}^{=1} + \frac{\partial f}{\partial z}\frac{dz}{dy} = 0$$

$$\frac{\partial g}{\partial x}\frac{dx}{dy} + \frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{dz}{dy} = 0$$

$$\frac{\partial f}{\partial x}\frac{dx}{dy} + \frac{\partial f}{\partial z}\frac{dz}{dy} = -\frac{\partial f}{\partial y}$$

$$\frac{\partial g}{\partial x}\frac{dx}{dy} + \frac{\partial g}{\partial z}\frac{dz}{dy} = -\frac{\partial g}{\partial y}$$

$$\Rightarrow \underbrace{\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial z} \end{pmatrix}}_{\text{Jacobian matrix}} \begin{pmatrix} \frac{dx}{dy} \\ \frac{dz}{dy} \end{pmatrix} = -\begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{dx}{dy} \\ \frac{dz}{dy} \end{pmatrix} = -J^{-1}\begin{pmatrix} \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial y} \end{pmatrix}$$

$$J^{-1} = \frac{1}{\text{Det } J}\begin{pmatrix} \frac{\partial g}{\partial z} & -\frac{\partial f}{\partial z} \\ -\frac{\partial g}{\partial x} & \frac{\partial f}{\partial x} \end{pmatrix}$$

Partial derivatives: $\frac{\partial f}{\partial x} = y^\alpha z$, $\frac{\partial g}{\partial x} = \beta x^{\beta-1} e^{yz}$, $\frac{\partial f}{\partial y} = \alpha x y^{\alpha-1} z$

$\frac{\partial f}{\partial z} = x y^\alpha$ $\quad \frac{\partial g}{\partial z} = x^\beta y e^{yz}$, $\frac{\partial g}{\partial y} = z x^\beta e^{yz}$

$$\text{Det } J = x^\beta y^{\alpha+1} z e^{yz} - \beta x^\beta y^\alpha e^{yz}$$

$$J^{-1} = \frac{1}{x^\beta y^{\alpha+1} z e^{yz} - \beta x^\beta y^\alpha e^{yz}}\begin{pmatrix} x^\beta y e^{yz} & -x y^\alpha \\ -\beta x^{\beta-1} e^{yz} & y^\alpha z \end{pmatrix}$$

$$\begin{pmatrix} dx/dy \\ dz/dy \end{pmatrix} = -\frac{1}{x^\beta y e^{yz}(zy-\beta)}\begin{pmatrix} x^\beta y e^{yz} & -x y^\alpha \\ -\beta x^{\beta-1} e^{yz} & y^\alpha z \end{pmatrix}\begin{pmatrix} \alpha x y^{\alpha-1} z \\ z x^\beta e^{yz} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} dx/dy \\ dz/dy \end{pmatrix} = \frac{-1}{x^{\beta} y\, e^{yz}(zy-\beta)} \begin{pmatrix} (\alpha-1)\, z x^{\beta+1} y^{\alpha} e^{yz} \\ (zy-\alpha\beta)\, x^{\beta} z y^{\alpha+1} e^{yz} \end{pmatrix}$$

$=$

<u>Please note</u>:

* The Determinant of the Jacobian matrix is called "the Jacobian".

* If $zy-\beta=0$ the Jacobian matrix is NOT invertible. Hence dx/dy, dz/dy do not exist in this case (e.g., for $z=\beta/y$)

$$dx/dy = \frac{-1}{x^{\beta} y e^{yz}(zy-\beta)} \cdot (\alpha-1)\, z x^{\beta+1} y^{\alpha} e^{yz} = \frac{-(\alpha-1)\, z x\, y^{\alpha-1}}{(zy-\beta)}$$

$$\frac{dz}{dy} = \frac{-1}{\cancel{x^{\beta}} y \cancel{e^{yz}}(zy-\beta)} (zy-\alpha\beta) \cancel{x^{\beta}} z\, y^{\alpha+1} \cancel{e^{yz}} = \frac{-(zy-\alpha\beta) z y^{\alpha}}{(zy-\beta)}$$

5 $f(x,y) = 1 - y^3 - 3yx^2 - 3y^2 - 3x^2$

- Extrema: $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$

(*) $\frac{\partial f}{\partial x} = -6yx - 6x = 0 \Rightarrow -6yx - 6x = 0 \quad \div -6 \Rightarrow yx + x = 0$

$x(y+1) = 0$

$x = 0$
or $y = -1$

(**) $\frac{\partial f}{\partial y} = -3y^2 - 3x^2 - 6y = 0$

Inserting $x=0$ in (**) $\Rightarrow -3y^2 - 6y = 0$

$y^2 + 2y = 0 < \begin{matrix} y_1 = 0 \\ y_2 = -2 \end{matrix}$

Inserting $y = -1$ in (**) $= -3 - 3x^2 + 6 = 0$

$x^2 = 1 < \begin{matrix} x_1 = 1 \\ x_2 = -1 \end{matrix}$

Critical points: $(x_0, y_0, f(x_0,y_0))$

$P_1: (0,0,1)$, $P_2: (0,-2,-3)$

$P_3: (1,-1,-1)$, $P_4: (-1,-1,-1)$

- Classification:

Hessian matrix $D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

$\frac{\partial^2 f}{\partial x^2} = -6y - 6 \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -6x$

$\frac{\partial^2 f}{\partial y^2} = -6y - 6$

$D^2 f = \begin{pmatrix} -6y-6 & -6x \\ -6x & -6y-6 \end{pmatrix}$

$P_1: (0,0,1) \Rightarrow D^2 f = \begin{pmatrix} -6 & 0 \\ 0 & -6 \end{pmatrix}$

NEGATIVE DEFINITE: P_1 is a maximum

@ $P_2 \Rightarrow D^2 f = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}$ Positive definite: P_2 is a minimum

@ $P_3 \Rightarrow D^2 f = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}$ $\mathrm{Det}[D^2 f] = -36$

P_3 is a saddle point

@ $P_4 \Rightarrow D^2 f = \begin{pmatrix} 0 & 6 \\ 6 & 0 \end{pmatrix}$ $\mathrm{Det}[D^2 f] = -36$

P_4 is a saddle point