Foundations of Numerical Methods
(2^{nd} term 2005)

Exercise Sheet 7 - Partial Differential Equations

1. Use the finite difference method to approximate the solution to

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 4, \qquad 0 < x < 1, 0 < y < 2$$

with the boundary conditions

$$\begin{aligned} u(x,0) &= x^2, u(x,2) = (x-2)^2, & 0 \le x \le 1 \\ u(0,y) &= y^2, u(1,y) = (y-1)^2, & 0 \le y \le 2 \end{aligned}$$

Use $\Delta x = \Delta y = 1/2$ and compare the results with the actual solution $u(x,y) = (x-y)^2$.

Solution:

i	j	x_i	y_j	$w_{i,j}$	$u(x_i, y_j)$
1	1	0.5	0.5	0.0	0
1	2	0.5	1.0	0.25	0.25
1	3	0.5	1.5	1.0	1

2. Approximate the solution to the following partial differential equation using the Crank-Nicolson method

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - \frac{1}{16}\frac{\partial^2 u(x,t)}{\partial x^2} &= 0, \qquad 0 < x < 1, \qquad t > 0 \\ u(0,t) &= u(1,t) = 0, \qquad t > 0, \ \ u(x,0) = 2\sin(2\pi x), \qquad 0 \le x \le 1. \end{aligned}$$

Use $m = 3, T_{\max} = 0.1$ and compare your results with the actual solution $u(x,t) = 2\exp[-\pi^2/4t]\sin(2\pi x)$

Solution:

i	j	x_i	t_j	w_{ij}
1	1	0.5	0.05	0.628848
2	1	1.0	0.05	0.889326
3	1	1.5	0.05	0.628848
1	2	0.5	0.1	0.559251
2	2	1.0	0.1	0.790901
3	2	1.5	0.1	0.559252

3. Suppose we wish to solve the parabolic equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} = h(x)$$

Derive the explicit forward difference equation, the backward implicit difference equation and the semi-implicit (within the Crank Nicolson scheme) difference equation for this situation.