(1)

(a) $f(x,y) = x^{\alpha} y^{\beta}$

• Gradient : $Df = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$ $\quad \frac{\partial f}{\partial x} = \alpha x^{\alpha-1} y^{\beta}$, $\quad \frac{\partial f}{\partial y} = \beta x^{\alpha} y^{\beta-1}$ $\Rightarrow Df = \begin{pmatrix} \alpha x^{\alpha-1} y^{\beta} \\ \beta x^{\alpha} y^{\beta-1} \end{pmatrix}$

• Hessian matrix : $D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \alpha(\alpha-1) x^{\alpha-2} y^{\beta}$; $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \beta(\beta-1) x^{\alpha} y^{\beta-2}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \alpha\beta x^{\alpha-1} y^{\beta-1}$; $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \alpha\beta x^{\alpha-1} y^{\beta-1}$

$$D^2 f = \begin{pmatrix} \alpha(\alpha-1) x^{\alpha-2} y^{\beta} & \alpha\beta x^{\alpha-1} y^{\alpha-1} \\ \alpha\beta x^{\alpha-1} y^{\beta-1} & \beta(\beta-1) x^{\alpha} y^{\beta-2} \end{pmatrix}$$

(b) $f(x,y) = x^2 \sin(2xy)$

Gradient : $Df = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$ $\quad \frac{\partial f}{\partial x} = 2x \sin(2xy) + 2x^2 y \cos(2xy)$, $\quad \frac{\partial f}{\partial y} = 2x^3 \cos(2xy)$

Hessian matrix : $D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

$\frac{\partial^2 f}{\partial x^2} = 2\sin(2xy) + 4xy\cos(2xy) + 4xy\cos(2xy) - 4x^2y^2\sin(2xy)$

$\frac{\partial^2 f}{\partial y^2} = -4x^4 \sin(2xy)$

(2)

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6x^2 \cos(2xy) + 4yx^3 \sin(2xy)$$

$$D^2 f = \begin{pmatrix} 2\sin(2xy) + 8xy\cos(2xy) - 4x^2y^2 \sin 2xy & 6x^2 \cos 2xy - 4yx^3 \sin 2xy \\ 6x^2 \cos(2xy) - 4yx^3 \sin(2xy) & -4x^4 \sin(2xy) \end{pmatrix}$$

CORRECT SOLUTION DISCUSSED IN CLASS

(c) $f(x,y) = e^{3xy^2}(x^2+1)$

- Gradient

$$Df = \begin{pmatrix} \partial f / \partial x \\ \partial f / \partial y \end{pmatrix}$$

with $\frac{\partial f}{\partial x} = 3y^2 e^{3xy^2}(x^2+1) + 2x e^{3xy^2}$

$\frac{\partial f}{\partial y} = 6xy e^{3xy^2}(x^2+1)$

- Hessian matrix

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

with $\frac{\partial^2 f}{\partial x^2} = (3y^2)^2 e^{3xy^2}(x^2+1) + 6xy^2 e^{3xy^2} +$

$+ 2e^{3xy^2} + 6xy^2 e^{3xy^2}$

$\frac{\partial^2 f}{\partial x^2} = e^{3xy^2}(9y^4(x^2+1) + 12xy^2 e^{3xy^2} + 2)$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6y e^{3xy^2}(x^2+1) + 18xy^3 e^{3xy^2}(x^2+1) + 12x^2 y e^{3xy^2}$$

$$\frac{\partial^2 f}{\partial y^2} = 6x e^{3xy^2}(x^2+1) + 36x^2y^2 e^{3xy^2}(x^2+1)$$

2.

Differential : $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

Total derivatives: $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$

1.(a) : $df = \alpha x^{\alpha-1} y^\beta dx + \beta x^\alpha y^{\beta-1} dy$

$\frac{df}{dx} = \alpha x^{\alpha-1} y^\beta + \beta x^\alpha y^{\beta-1} \frac{dy}{dx}$

Since $y = 2x$, $\frac{df}{dx} = \alpha x^{\alpha-1}(2x)^\beta + \beta x^\alpha (2x)^{\beta-1} \cdot 2$

$= 2^\beta \alpha x^{\alpha+\beta-1} + 2^\beta \beta x^{\alpha+\beta-1} = 2^\beta(\alpha+\beta) x^{\alpha+\beta-1}$

3

1(b) $df = \left(2x\sin(2xy) + 2x^2y\cos(2xy)\right)dx + 2x^3\cos(2xy)\,dy$

$$\frac{df}{dx} = \left(2x\sin(2xy) + 2x^2y\cos(2xy)\right)\frac{dx}{dx} + 2x^3\cos(2xy)\frac{dy}{dx}$$

$y = 2x$

$$\boxed{\begin{aligned}\frac{df}{dx} &= 2x\sin(4x^2) + 4x^3\cos(4x^2) + 2x^3\cos(4x^2)\cdot 2x \\ &= 2x\sin(4x^2) + 8x^3\cos(4x^2)\end{aligned}}$$

3

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} =$$

$x = e^t,\ y = 1+t$

$\frac{dx}{dt} = e^t$

$\frac{\partial f}{\partial x} = y,\ \frac{\partial f}{\partial y} = x + 2y \qquad \frac{dy}{dt} = 1$

$$\Rightarrow \boxed{\frac{df}{dt} = (1+t)e^t + e^t + 2(1+t)}$$

4

4 $Q(k,L) = 20k^{1/2}L^{1/3}$

a) $\frac{\partial Q}{\partial k} = 10k^{-1/2}L^{1/3}$; $\frac{\partial Q}{\partial L} = \frac{20}{3}k^{1/2}L^{-2/3}$

(**) $k\frac{\partial Q}{\partial k} = 10k^{1/2}L^{1/3}$; $L\frac{\partial Q}{\partial L} = \frac{20}{3}k^{1/2}L^{1/3}$ (*)

$(*)+(**) = \left(\frac{20}{2}+\frac{20}{3}\right)k^{1/2}L^{1/3}$

$= 20\underbrace{\left(\frac{1}{2}+\frac{1}{3}\right)}_{\frac{5}{6}}k^{1/2}L^{1/3} \Rightarrow \boxed{C = \frac{5}{6}}$

Gradient:

$$DQ = \begin{pmatrix} \frac{20}{2}k^{-1/2}L^{1/3} \\ \\ \frac{20}{3}k^{1/2}L^{-2/3} \end{pmatrix}$$

(c) Q is homogeneous

$Q(\lambda k, \lambda L) = 20\lambda^{1/2+1/3}k^{1/2}L^{1/3}$

(Degree $= 5/6$)

According to Euler's theorem, for an homogeneous function

$k\frac{\partial Q}{\partial k} + L\frac{\partial Q}{\partial L} = rQ$, where r is the degree of the function. $r = 5/6$

5

(a) $f(x,y) = x^3 - xy + y^3$

$f(\lambda x, \lambda y) = \lambda^3 x^3 - \lambda^2 xy + \lambda^3 y^3$ No

(b) $f(x,y) = 2x + y + 3\sqrt{xy}$

$f(\lambda x, \lambda y) = 2\lambda x + 2\lambda y + 3\lambda\sqrt{xy} = \lambda f(x,y)$

Yes (Homogeneous of degree 1)

(c) $f(x,y) = \exp[x^2 + y^2]$

$f(\lambda x, \lambda y) = \exp[\lambda^2 x^2 + \lambda^2 y^2] = [f(x,y)]^{\lambda^2}$ No