Exercise sheet 6 - further maths - solutions

(1)

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

• General solution

Setting y of the form $y = e^{\alpha t}$ we get

$$(\alpha^2 + 5\alpha + 6)\, e^{\alpha t} = 0$$

$$\alpha = -\frac{5}{2} \pm \frac{\sqrt{25-24}}{2} \quad \alpha_1 = -2, \quad \alpha_2 = -3$$

$$\boxed{y = Ae^{-3t} + Be^{-2t}}$$

• Specific solution

$y(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

$\left.\frac{dy}{dt}\right|_{t=0} = 1 \Rightarrow -3A - 2B = 1$

$3B - 2B = 1 \Rightarrow B = 1$

$A = -1$

$$\boxed{y = -e^{-3t} + e^{-2t}}$$

(2)

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 0$$

• General solution: $y = Ae^{\alpha_1 t} + Be^{\alpha_2 t}$ where α_1, α_2 obtained from the characteristic equation

$$\alpha^2 + 6\alpha + 10 = 0$$

$$\alpha = -3 \pm \frac{\sqrt{36-40}}{2} \quad \alpha_1 = -3 + \frac{\cancel{2}i}{\cancel{2}}, \quad \alpha_2 = -3 - i$$

$$y = Ae^{-3t+it} + Be^{-3t-it} = e^{-3t}\left(Ae^{it} + Be^{-it}\right)$$

- Specific solution:

$$y(0) = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$$

$$y(\pi/2) = 1 \Rightarrow Ae^{-3\pi/2}\left(e^{i\pi/2} - e^{-i\pi/2}\right) = 1$$

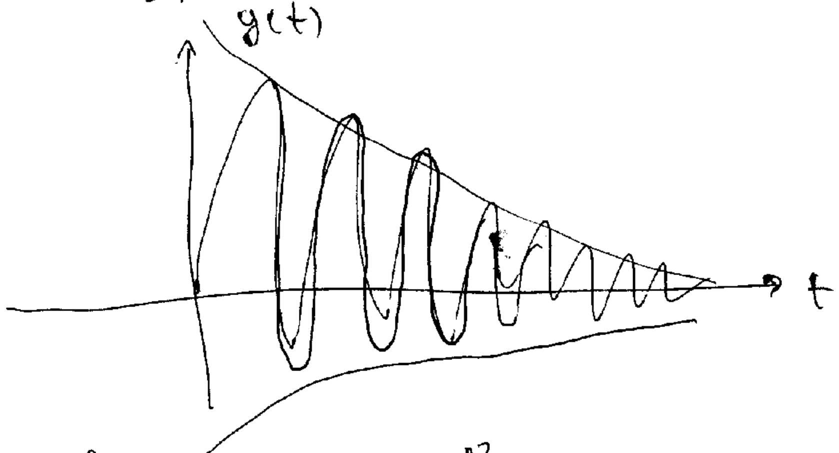
But $e^{i\pi/2} - e^{-i\pi/2} = 2i \sin\frac{\pi}{2} = 2i$

Hence $A = \dfrac{e^{3\pi/2}}{2i}$

$B = -\dfrac{e^{3\pi/2}}{2i}$

$$\boxed{y = e^{-3(t-\pi/2)}\left(\frac{1}{2i}e^{it} - \frac{1}{2i}e^{-it}\right) = e^{-3(t-\pi/2)}\sin t}$$

(*) Qualitative behavior: this solution corresponds to damped oscillations.



(3) • Find the general solution of $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4 = 0$

The characteristic equation of this solution is

$$\alpha^2 + 4\alpha + 4 = 0$$

This equation has coincident roots at $\alpha = 2$. We have then that $y = Ae^{2t} + Bte^{2t}$ is the general solution

(You can check that Bte^{2t} is a solution)

• Specific solution: $y(0) = 1 \Rightarrow A = 1$

$y(1) = 0 \Rightarrow A e^{-2} + B e^{-2} = 0$

$A = -B$

$\Rightarrow A = 1,\ B = -1$

$$\boxed{y = e^{-2t} - t e^{-2t}}$$

(4) (a) $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 5y = 3e^t + 5 \quad (*)$

• Particular solution: $y_P = a e^t + b$

$\dfrac{dy_P}{dt} = a e^t$

$\dfrac{d^2y_P}{dt^2} = a e^t$

Inserting into $(*)$: $(\underbrace{7a + 5a}_{12a})e^t + 5b = 3e^t + 5$

$y_P = \dfrac{e^t}{4} + 1$ $\qquad 12a = 3 \Rightarrow a = 1/4$

$5b = 5 \qquad b = 1$

• Complementary solution:

$\dfrac{d^2z}{dt^2} + 6\dfrac{dz}{dt} + 5z = 0 \quad (*)$

Ansatz: $z = e^{\alpha t}$ so that $(*) = (\alpha^2 + 6\alpha + 5)e^{\alpha t} = 0$

$\alpha_1 = -3 - \frac{1}{2}\sqrt{36 - 20} = -5$

$\alpha_2 = -3 + 2 = -1$

$z = A e^{-5t} + B e^{-t}$

$$\boxed{y = A e^{-5t} + B e^{-t} + \frac{e^t}{4} + 1}$$

(b) $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 5 = 3e^{-t}$ $(*)$

- Particular solution. $y_P = ate^{-t}$ (otherwise it will cancel out)

$$\frac{dy_P}{dt} = ae^{-t} - ate^{-t} \quad ; \quad \frac{d^2y_P}{dt^2} = -ae^{-t} - ae^{-t} + ate^{-t}$$
$$= -2ae^{-t} + ate^{-t}$$

Substituting in $(*)$:

$$-2ae^{-t} + \cancel{ate^{-t}} + 6ae^{-t} - \cancel{6ate^{-t}} + \cancel{5ate^{-t}} = 3e^{-t}$$
$$\Rightarrow (6a - 2a)e^{-t} = 3e^{-t}$$
$$4ae^{-t} = 3e^{-t} \Rightarrow a = 3/4$$
$$y_P = \frac{3}{4}te^{-t}$$

- Complementary solution:

$$\frac{d^2z}{dt^2} + 6\frac{dz}{dt} + 5 = 0 \quad \text{with } z = y - y_P$$

(the same as in (a)):

$$z = Ae^{-5t} + Be^{-t}$$

$$\boxed{y = z + y_P = \frac{3}{4}te^{-t} + Ae^{-5t} + Be^{-t}}$$

(c) $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 26t + 21$

- Particular solution: $y_P = at + b$

$$\frac{dy_P}{dt} = a \quad ; \quad \frac{d^2y_P}{dt^2} = 0 \qquad 4a + 13at + 13b = 26t + 21$$

$$13a\cancel{t} = 26 \Rightarrow a = 2$$
$$4a + 13b = 21$$
$$13b = 21 - 8 = 13$$
$$b = 1$$

$y_p = 2t + 1$

- Complementary solution: $\frac{d^2 z}{dt^2} + 4\frac{dz}{dt} + 13z = 0$ with $z = y - y_p$

characteristic equation (obtained assuming z of the form $z = e^{\alpha t}$)

$$\alpha^2 + 4\alpha + 13 = 0$$

$$\alpha = -2 \pm \frac{1}{2}\sqrt{16 - 52} = -2 \pm \frac{1}{2}\sqrt{-36} = -2 \pm \frac{6}{2}i$$

$$\alpha_1 = -2 + 3i$$

$$\alpha_2 = -2 - 3i$$

$$z = e^{-2t}\left(A e^{3it} + B e^{-3it}\right)$$

$$\boxed{y = 2t + 1 + e^{-2t}\left(A e^{3it} + B e^{-3it}\right)}$$