Foundations of Numerical Methods
(2^{nd} term 2005)

Exercise Sheet 6 - Interpolation/Numerical Integration

1. Neville's method is used to approximate $f(0.5)$ giving the following table:

$$\begin{array}{llll} x_0 = 0 & P_0 = 0 & & \\ x_1 = 0.4 & P_1 = 2.8 & P_{0,1} = 3.5 & \\ x_2 = 0.7 & P_2 & P_{1,2} & P_{0,1,2} = 27/7 \end{array}$$

   Find $P_{1,2}$ and P_2.

2. Determine the Padé approximation of degree six for $f(x) = \sin(x)$, with $\nu = \mu = 3$. Compare the results at $x_i = 0.2i$, for $i = 1, 2, 3$ with the actual values of $f(x_i)$ and with its sixth Taylor polynomial.

3. Construct a free cubic spline to approximate $f(x) = \cos \pi x$ by using the values given by $f(x)$ at $x = 0, 0.25, 0.5, 0.75$ and 1.0.

   (a) Integrate the spline over $[0, 1/2]$ and compare the result to $\int_0^{1/2} \cos \pi x dx = 1/\pi$

   (b) Use the first derivative of the spline to compute $f'(0.3)$ and $f'(0.5)$. Compare these approximations to the actual value.

4. A clamped cubic spline S on $[1, 3]$ is defined by

$$S(x) = \begin{cases} S_0(x) = 3(x-1) + 2(x-1)^2 - (x-1)^3, & \text{if } 1 \leq x \leq 2, \\ S_1(x) = a + b(x-2) + c(x-2)^2 + d(x-2)^2, & \text{if } 2 \leq x \leq 3. \end{cases}$$

   Given $f'(1) = f'(3)$, find a, b, c and d

5. A Fredholm integral equation of the second kind is an equation of the form

$$u(x) = f(x) + \int_a^b K(x,t)u(t)dt,$$

   where a and b and the functions f and K are given. To approximate the function u on the interval [a,b], a partition $x_0 = a < x_1 < ... < x_{m-1} < x_m = b$ is selected and one solves the equations

$$u(x_i) = f(x_i) + \int_a^b K(x_i,t)u(t)dt,$$

   for $i = 1, ..., m$. The integrals are approximated by quadrature formulae based on the nodes $x_0, x_1, ..., x_m$. In our problem, $a = 0, b = 1, f(x) = x^2$ and $K(x,t) = \exp[|x-t|]$.

(a) Show that the linear system

$$u(0) = f(0) + \frac{1}{2}\left[K(0,0)u(0) + K(0,1)u(1)\right], u(1) = f(0) + \frac{1}{2}\left[K(1,0)u(0) + K(1,1)u(1)\right]$$

must be solved when the Trapezoidal rule is used. Write this linear system explicitly

(b) Set up and solve the linear system that results when the Composite Trapezoidal rule is used with $n = 4$