Solutions - further maths for economists - sheet 5

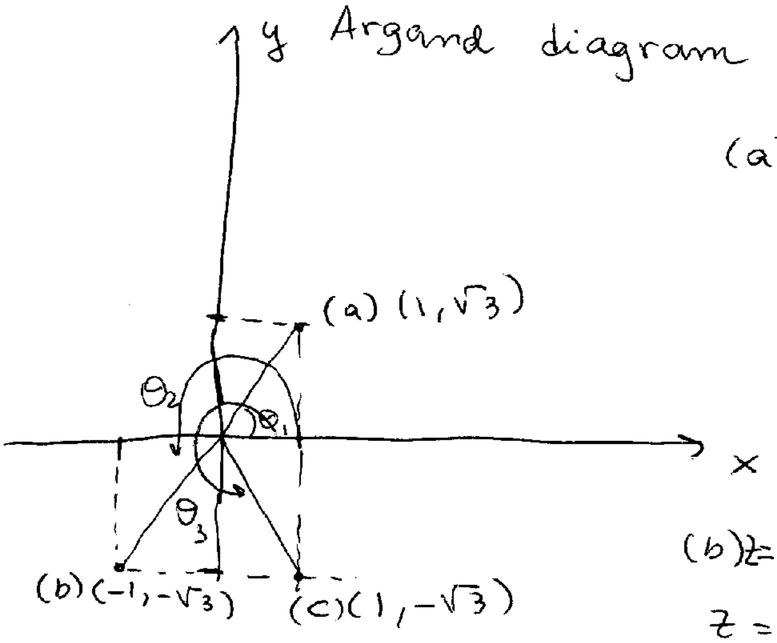
① $z_1 = 1+2i$; $z_2 = 2-i$

(a) $z_1 z_2 = (1+2i)(2-i) = 2 \overbrace{- i + 2i\cdot 2}^{3i} \underbrace{- i \cdot 2i}_{-2\cdot(-1)=2} = 4+3i$

(b) $z_1/z_2 = \dfrac{1+2i}{2-i} \cdot \dfrac{2+i}{2+i} = \dfrac{2+i+4i+\underbrace{2i^2}_{-2}}{\underbrace{4+1}_{5}} = \dfrac{5i}{5} = i$

(c) $= z_1 z_2^* = 5i$

② Argand diagram



(a) $z = 1 + i\sqrt{3}$ (algebraic form)

$z = \sqrt{4}(\cos\theta_1 + i\sin\theta_1)$ (trig. form)

$\theta_1 = \text{Arc}\tan(\sqrt{3}) = \frac{\pi}{3}$
(since $Re[z]$ and $Im[z] > 0$)

$z = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$

(b) $z = -1 - i\sqrt{3}$ (algebraic form)

$z = \sqrt{4}(\cos\theta_2 + i\sin\theta_2)$ (trig. form)

$\theta_2 = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$z = \sqrt{4}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$

Please note: $\theta_2 = \text{Arc}\tan(\sqrt{3})$, however, since $Re[z] < 0$ and $Im[z] < 0$, this angle should be in the 3rd quadrant

(c) $z = 1 - i\sqrt{3}$ (algebraic form)

$z = 2(\cos\theta_3 + i\sin\theta_3) = 2\Big(\cos\big(\overbrace{2\pi - \tfrac{\pi}{3}}^{5\pi/3}\big) + i\sin\big(\overbrace{2\pi - \tfrac{\pi}{3}}^{5\pi/3}\big)\Big)$

$\theta_3 = \text{Arc}\tan(-\sqrt{3}) = 2\pi - \frac{\pi}{3}$

3 $\Big(\underbrace{\frac{1}{2}+i\frac{\sqrt{3}}{2}}_{z_1}\Big)^{20} + \Big(\underbrace{\frac{1}{2}-i\frac{\sqrt{3}}{2}}_{z_2}\Big)^{20}$

we will apply de Moivre's theorem to find these powers
$\Rightarrow$ we need the trigonometric forms of z_1, z_2

<u>Trigonometric forms & de Moivre's theorem</u>

$$z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,(\cos\theta_1 + i\sin\theta_1)$$

$\theta_1 = \text{Arc}\tan\sqrt{3} = \frac{\pi}{3}$ (θ_1 is in the 1st quadrant since $Re[z_1] > 0$ & $Im[z_1] > 0$)

$$z_1 = 1\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) \Rightarrow z_1^{20} = 1^{20}\left(\cos\frac{20\pi}{3} + i\sin\frac{20\pi}{3}\right)$$

We can write $\frac{20\pi}{3} = \frac{18\pi}{3} + \frac{2\pi}{3} = 6\pi + \frac{2\pi}{3}$ (Note that 6π is a multiple of 2π)

This gives $z_1^{20} = 1\left[\cos\left(6\pi + \frac{2\pi}{3}\right) + i\sin\left(6\pi + \frac{2\pi}{3}\right)\right] = 1\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$
(since cos, sin are periodic in 2π)

But $\frac{2\pi}{3} = \pi - \frac{\pi}{3}$ so that $\sin\left(\frac{2\pi}{3}\right) = \sin\frac{\pi}{3}$
$\cos\left(\frac{2\pi}{3}\right) = -\cos\frac{\pi}{3}$ $\Rightarrow z_1^{20} = 1\left[-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

$$z_1^{20} = \left[-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]$$

$$z_2 = \frac{1}{2} - i\frac{\sqrt{3}}{2} = 1\,(\cos\theta_2 + i\sin\theta_2)$$

$\theta_2 = \text{Arc}\tan(-\sqrt{3})$ since $\cos\theta_2 > 0$ & $\sin\theta_2 < 0$, $\theta_2 = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$

and $z_2 = \left(\cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)\right)$

$$z_2^{20} = \cos\left(\frac{100\pi}{3}\right) + i\sin\left(\frac{100\pi}{3}\right) = \cos\left(\overbrace{\frac{96\pi}{3}}^{32\pi} + \frac{4\pi}{3}\right) + i\sin\left(\overbrace{\frac{96\pi}{3}}^{32\pi} + \frac{4\pi}{3}\right)$$
$$= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)$$

Since $\frac{4\pi}{3} = \pi + \frac{\pi}{3}$, $\sin\left(\frac{4\pi}{3}\right) = -\sin\left(\frac{\pi}{3}\right)$ $\Rightarrow z_2^{20} = \left(-\cos\left(\frac{\pi}{3}\right) - i\sin\frac{\pi}{3}\right)$
$\cos\left(\frac{4\pi}{3}\right) = -\cos\left(\frac{\pi}{3}\right)$ $\quad = -\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$$\Rightarrow \boxed{z_1^{20} + z_2^{20} = -\frac{1}{2} + \cancel{i\frac{\sqrt{3}}{2}} - \frac{1}{2} - \cancel{i\frac{\sqrt{3}}{2}} = -1}$$

(4) $z^4 = \underbrace{(8\sqrt{3}+8i)}_{(*)} \Rightarrow z = (8\sqrt{3}+8i)^{1/4}$

Trigonometric form of $(*) = \underbrace{\sqrt{(8\sqrt{3})^2+(8)^2}}_{8\times 2=16=2^4}\underbrace{(\cos\theta + i\sin\theta)}_{e^{i\theta}}$, with

$\theta = \text{Arc}\tan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} + \underbrace{2\pi n}$

remember: sin, cos are periodic in 2π.

$z = 2^{4\times 1/4}\left(e^{i\frac{\pi}{6}+i2\pi n}\right)^{1/4} = 2e^{i\frac{\pi}{24}}\cdot e^{i\frac{\pi n}{2}}$

$n=0 \Rightarrow z_0 = 2e^{i\pi/24} = 2\left[\cos\frac{\pi}{24} + i\sin\frac{\pi}{24}\right]$

$n=1 \Rightarrow z_1 = 2e^{i\pi/24+i\pi/2} = 2\underbrace{e^{i\pi/2}}_{i}\left[\cos\frac{\pi}{24}+i\sin\frac{\pi}{24}\right]$

$n=2 \Rightarrow z_1 = 2e^{i\pi/24+i\pi} = 2\underbrace{e^{i\pi}}_{-1}\left[\cos\frac{\pi}{24}+i\sin\frac{\pi}{24}\right]$

$n=3 \Rightarrow z_3 = 2e^{i\pi/24}\cdot e^{i3\pi/2} = -i\times 2e^{i\pi/24} = -2i\left[\cos\frac{\pi}{24}+i\sin\frac{\pi}{24}\right]$

$\left(\cos\frac{\pi}{24} = 0.9915 ; \sin\frac{\pi}{24} = 0.1305\right)$

(5) $(z^2 + 4z + 4) = 1 - i$ can be written as

$(z-2)^2 = 1 + i$ so that $(z-2) = (1-i)^{1/2}$

$z = 2 + (1-i)^{1/2}$

We have to find $\underbrace{(1-i)}^{1/2}$

$z_1 = (1-i) = \sqrt{2}(\cos\theta + i\sin\theta) = \sqrt{2}e^{i\theta}$

$\theta = \arctan(-1) = \underbrace{2\pi - \frac{\pi}{4}}_{\frac{7\pi}{4}} + 2n\pi$

$z_1^{1/2} = 2^{1/4}\cdot\left(e^{i\frac{7\pi}{4}+i2n\pi}\right)^{1/2}$

$n=0 \Rightarrow 2^{1/4}e^{i7\pi/8}$

$n=1 \Rightarrow 2^{1/4}e^{i7\pi/8}\cdot\underbrace{e^{i\pi}}_{-1}$

So the above-stated equation has the roots

$z_1 = 2 + 2^{1/4}e^{i7\pi/8}$

$z_2 = 2 - 2^{1/4}e^{i7\pi/8}$

(6)
$$\int e^{-x}\cos(2x)\,dx = \int e^{-x}\left(\frac{e^{2ix}+e^{-2ix}}{2}\right)dx = \int \frac{e^{-x}\cdot e^{2ix}}{2}\,dx + $$
$$+\frac{1}{2}\int \frac{e^{-x}e^{-2ix}}{2}\,dx$$

$$\int e^{-x}e^{2ix}\,dx = \frac{1}{-1+2i}e^{-x}e^{2ix} + c_1$$

$$\int e^{-x}e^{-2ix}\,dx = \int e^{-(\cancel{1}+2i)x}\,dx = \frac{1}{-(1+2i)}e^{-x}e^{-2ix} + c_2$$

$$\Rightarrow \int e^{-x}\cos(2x)\,dx = \frac{1}{2}\left[\frac{e^{-x}e^{2ix}}{-1+2i} + \frac{e^{-x}e^{-2ix}}{-(1+2i)}\right] + C$$

$$= \frac{e^{-x}}{2}\left[\frac{(1+2i)e^{2ix}+(1-2i)e^{-2ix}}{\underbrace{(-1)^2}_{1}\underbrace{[(1-2i)(1+2i)]}_{1+4}}\right] + c = \frac{e^{-x}}{5}\left[\left(\frac{e^{2ix}+e^{-2ix}}{2}\right) + 2i\left(\frac{e^{2ix}-e^{-2ix}}{2}\right)\right] + c$$

Note that

$$2i\left(\frac{e^{2ix}-e^{-2ix}}{2}\right) = -2\sin 2x$$

$$\frac{e^{2ix}+e^{-2ix}}{2} = \cos 2x \qquad \Rightarrow \boxed{\int e^{-x}\cos 2x\,dx = \frac{e^{-x}}{5}[\cos 2x - 2\sin 2x] + c}$$

(7) Starting point $z^2 + a_1 z + a_2 = 0$ (*)

We want to prove that if a_1, a_2 real then the roots of the above equation are a conjugate pair

(*) can be written as $(z-z_1)(z-z_2) = z^2 - (z_1+z_2)z + z_1 z_2$ (from the fundamental theorem of algebra)

So that $a_1 = -(z_1+z_2)$; $a_2 = z_1 z_2$

Using $z_1 = |z_1|e^{i\theta_1}$, $z_2 = |z_2|e^{i\theta_2}$ we have

$a_1 = -(|z_1|e^{i\theta_1} + |z_2|e^{i\theta_2})$ (a), $a_2 = |z_1||z_2|e^{i(\theta_1+\theta_2)}$ (b)

Writing (b) as $a_2 = |z_1||z_2|\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$ we know that, since a_2 is real, $\sin(\theta_1+\theta_2)=0$ $[\mathrm{Im}[a_2]=0)$

Hence $\theta_1+\theta_2 = n\pi \Rightarrow \theta_1 = -\theta_2 + n\pi$

This gives $a_1 = -\left(|z_1| e^{-i\theta_2+in\pi} + |z_2| e^{i\theta_2}\right) = -\left(|z_1|(-1)^n e^{-i\theta_2} + \right.$
$\left. + (|z_2| e^{i\theta_2}\right)$

n even $\Rightarrow$ $a_1 = -\left(|z_1| e^{-i\theta_2} + |z_2| e^{i\theta_2}\right)$ (c) $(\theta_1 = -\theta_2 + 2n\pi$

n odd $\Rightarrow$ $a_1 = -\left(-|z_1| e^{-i\theta_2} + |z_2| e^{i\theta_2}\right)$ (d) $(\theta_1 = -\theta_2 + (2n+1)\pi)$

(+) Condition (c)

$$a_1 = \mathrm{Re}[a_1] + i\underbrace{\mathrm{Im}[a_1]}_{=0\ (a_1 \text{ real})} = -\left(|z_1|(\cos\theta_2 - i\sin\theta_2) + |z_2|(\cos\theta_2 + i\sin\theta_2)\right)$$
$$= -\left[(|z_1|+|z_2|)\cos\theta_2 + \left(|z_2|-|z_1|\right)\cdot i\sin\theta_2\right]$$

From $\mathrm{Im}[a_1]=0$ we have
$(|z_1|-|z_2|)\sin\theta_2 = 0$ (satisfied if $|z_1|=|z_2|$)

(+) Condition (d):

$$a_1 = \mathrm{Re}[a_1] + i\underbrace{\mathrm{Im}[a_1]}_{=0} = -\left(-|z_1|(\cos\theta_2 - i\sin\theta_2) + |z_2|(\cos\theta_2 + i\sin\theta_2)\right)$$
$$= -\left((|z_2|-|z_1|)\cos\theta_2 + i\left(|z_1|+|z_2|\right)\sin\theta_2\right]$$

$\mathrm{Im}[a_1]=0 \Rightarrow (|z_1|+|z_2|)\sin\theta_2 = 0$ can only be satisfied if $\sin\theta_2 = 0$
since $|z_1|>0$, $|z_2|>0$ $\Rightarrow$ ~~does not~~ should be discarded, since our condition should hold $\forall\, \theta_1, \theta_2$.

Hence $z_1 z_2$ are such that $\theta_1 = -\theta_2 + 2n\pi$
$|z_1| = |z_2|$

$$z_1 = |z_2| e^{-i\theta_2 + i2n\pi} = |z_2| e^{-i\theta_2}$$
$$z_2 = |z_2| e^{i\theta_2} \Rightarrow \boxed{z_1 = z_2^*}$$