Further mathematics for economists

Exercise Sheet 6 - Complex numbers

1. Consider the complex numbers $z_1 = 1 + 2i$ and $z_2 = 2 - i$ and compute

   (a) $z_1 z_2$

   (b) z_1/z_2

   (c) $z_1 z_2^*$

2. Write the algebraic and trigonometric forms for the following points in an Argand diagram

   (a) $(1, \sqrt{3})$

   (b) $(-1, -\sqrt{3})$

   (c) $(1, -\sqrt{3})$

3. Compute $(1/2 + i\sqrt{3}/2)^{20} + \left(1/2 - i\sqrt{3}/2\right)^{20}$ using de Moivre's theorem

4. Find all the roots of $z^4 = (8\sqrt{3} + 8i)$

5. Solve the quadratic equation $z^2 - 4z + 4 = 1 - i$ (hint: factorize the LHS and compute a square root)

6. Compute $\int e^{-x} \cos(2x)dx$ (hint: note that $\cos(u) = \frac{1}{2}(e^{iu} + e^{-iu})$)

7. Prove that, if the coefficients $a_k (k = 1, 2)$ in $z^2 + a_1 z + a_2 = 0$ are real, then the roots of this equation are a conjugate pair.
   Hints:

   - Factorize the equation as $(z - z_1)(z - z_2) = z^2 - (z_1 + z_2)z + z_1 z_2$, where $z_1 z_2$ are the roots of the equation
   - Use $z_1 = |z_1|e^{i\theta_1}$ and $z_2 = |z_2|e^{i\theta_2}$
   - Show that $a_1 = -(z_1+z_2)$ and $a_2 = z_1 z_2$ can only be real if $|z_1| = |z_2|$ and if $\theta_2 = -\theta_1+2n\pi$, for arbitrary θ (you will need the Euler formula $e^{i\theta} = \cos\theta + i \sin\theta$, and the conditions $\text{Im}[a_1] = 0$ and $\text{Im}[a_2] = 0$)
   - Note that $e^{in\pi} = (-1)^n$.