1. $\frac{dy}{dt} + 2y = 3e^{-t}$ is an inhomogeneous linear differential equation of const. coefficients

To solve this equation we must:

(a.) Find the particular solution:

Given: $y_p = Ce^{-t}$

$$\Rightarrow \frac{dy_p}{dt} + 2y_p = -Ce^{-t} + 2Ce^{-t} = 3e^{-t} \Rightarrow C = 3$$

and $y_p = 3e^{-t}$

(b) Find the complementary solution; i.e., the solution of $\frac{dz}{dt} + 2z = 0$, with $z = y - y_p$

$$\Rightarrow z(t) = Ae^{-2t}$$

So that $y(t) = Ae^{-2t} + 3e^{-t}$

Initial condition: $y(0) = 4$

$y(0) = A + 3 = 4 \Rightarrow A = 1$

$$\boxed{y(t) = e^{-2t} + 3e^{-t}}$$

2.

(a) $x\frac{dy}{dx} - 2y = x^5$ (*)

This differential equation will be solved by finding an integrating factor

- Dividing (*) by x we have $\frac{dy}{dx} - \frac{2}{x}y = x^4$ (**)

- Finding the integrating factor we have:

$(f(x) = e^{-\int h(x)dx}$, with $h(x) = -\frac{2}{x}$

$$f(x) = e^{-\int \frac{2}{x} dx} = e^{-2\ln|x|} = e^{\ln\left|\frac{1}{x^2}\right|} = \frac{1}{x^2}$$

- Multiplying (**) by $f(x)$ we have

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2}{x^3} y = x^2 \Rightarrow \frac{d}{dx}\left(\frac{1}{x^2} y(x)\right) = x^2$$

$$\Rightarrow \frac{1}{x^2} y(x) = \int x^2 dx = \frac{x^3}{3} + C \Rightarrow \boxed{y = \frac{x^5}{3} + Cx^2}$$

(b) $(x^2+1)\frac{dy}{dx} + xy = 1$ (*)

- Dividing (*) by $(1+x^2)$ we have $\frac{dy}{dx} + \frac{x}{x^2+1} y = \frac{1}{x^2+1}$ (**)

- The integrating factor will be

$$f(x) = e^{\int \frac{x}{x^2+1} dx}$$

The integral $I = \int \frac{x\,dx}{x^2+1}$ will be solved by substitution:

$$u = x^2+1$$
$$du = 2x\,dx \Rightarrow I = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u|$$

$$\Rightarrow f(x) = e^{\frac{1}{2}\ln|1+x^2|} = \sqrt{1+x^2}$$

- Multiplying (**) by $f(x)$ we have

$$\sqrt{1+x^2}\frac{dy}{dx} + \frac{x}{\sqrt{1+x^2}} y = \frac{1}{\sqrt{x^2+1}} \Rightarrow \frac{d}{dy}\left(\sqrt{1+x^2}\, y(x)\right) = \frac{1}{\sqrt{1+x^2}}$$

$$\Rightarrow \sqrt{1+x^2}\, y(x) = \underbrace{\int \frac{dx}{\sqrt{x^2+1}}}_{I}$$

$$I = \int \frac{dx}{\sqrt{x^2+1}} \cdot \frac{(x+\sqrt{x^2+1})}{(x+\sqrt{x^2+1})} = \int \frac{\left(1 + \frac{x}{\sqrt{x^2+1}}\right) dx}{x+\sqrt{x^2+1}} = \ln|x+\sqrt{x^2+1}| + C$$

$$\Rightarrow \sqrt{1+x^2}\, y(x) = \ln\left|x+\sqrt{x^2+1}\right| + C$$

$$\boxed{y(x) = \frac{1}{\sqrt{1+x^2}}\left[\ln\left|x+\sqrt{x^2+1}\right| + C\right]}$$

(c) $2\frac{dy}{dt} + y + (1+t)y^3 = 0 \Rightarrow \frac{dy}{dt} + \frac{y}{2} = -\frac{(1+t)}{2}y^3 \quad (*)$

dividing by $2(1+t)$ and rearranging

Since $(*)$ is a Bernoulli equation we will make the following change of variable:

$$x = y^{-2} \Rightarrow \frac{dx}{dt} = -2y^{-3}\frac{dy}{dt} = -2y^{-3}\left(-\frac{y}{2} - \frac{(1+t)}{2}y^3\right)$$

$$\Rightarrow \frac{dx}{dt} = \underbrace{y^{-2}}_{x} + (1+t) \quad (**)$$

Rearranging $(**)$ we have $\frac{dx}{dt} - x = (1+t) \quad (***)$

This is a linear 1^{st}-order inhomogeneous differential equation of constant coefficients: $x(t) = x_p + z$, where x_p is the particular solution and z satisfies $\frac{dz}{dt} - z = 0$

- Particular solution: $x_p = a + bt$

Inserting in $(***) \Rightarrow$ $b - (a+bt) = 1 + t$

$b - a = 1 \quad a = -2$

$-b = 1 \Rightarrow b = -1$

$x_p = -2 - t$

- Complementary solution: $z(t) = ?$ with $\frac{dz}{dt} - z = 0 \Rightarrow$

$\Rightarrow z(t) = Ae^t \Rightarrow x(t) = Ae^t + x_p = Ae^t - 2 - t$

but $x = y^{-2}$ so that $y = \pm \frac{1}{\sqrt{x}} \Rightarrow \boxed{y(t) = \pm \frac{1}{\sqrt{Ae^{t} - 2 - t}}}$