Further mathematics for economists - exercise sheet 3 - solutions

1. (a) $\frac{2}{t}\frac{dx}{dt} + x^2\sqrt{a-t^2} = 0$

Multiplying the above equation by $\frac{t}{2}$, we obtain

$$\frac{dx}{dt} + \frac{x^2 t}{2}\sqrt{a-t^2} = 0 \Rightarrow \frac{dx}{dt} = \underbrace{-\frac{x^2}{2}t\sqrt{a-t^2}}$$

the RHS is of the form $F(x)G(t)$: the above differential eq. can be solved by separation of variables

$$\frac{1}{x^2}\frac{dx}{dt} = -\frac{t}{2}\sqrt{a-t^2} \Rightarrow \underbrace{\int \frac{1}{x^2}dx}_{I_1} = \underbrace{-\int \frac{t}{2}\sqrt{a-t^2}\,dt}_{I_2}$$

$I_1 = -\frac{1}{x} + C_1$ (power rule)

$$I_2 = -\frac{1}{2}\int t\sqrt{a-t^2}\,dt = -\frac{\sqrt{a}}{2}\int t\sqrt{1-\left(\frac{t}{\sqrt{a}}\right)^2}\,dt$$

By substitution: $1-\left(\frac{t}{\sqrt{a}}\right)^2 = u \Rightarrow \frac{du}{dt} = -\frac{2t}{a} \Rightarrow t\,dt = -\frac{a\,du}{2}$

$$\Rightarrow I_2 = \frac{a^{3/2}}{4}\int \sqrt{u}\,du = \frac{a^{3/2}}{\cancel{4}\,2} \times \frac{\cancel{2}}{3}u^{3/2} + C_2 = \frac{a^{3/2}}{6}\sqrt{\left(1-\left(\frac{t}{\sqrt{a}}\right)^2\right)^3} + C_2$$

$$\Rightarrow -\frac{1}{x} + C_1 = \frac{a^{3/2}}{6}\sqrt{\left(1-\left(\frac{t}{\sqrt{a}}\right)^2\right)^3} + C_2$$

$$-\frac{1}{x} = \frac{a^{3/2}}{6}\sqrt{\left(1-\left(t/\sqrt{a}\right)^2\right)^3} + C$$

$$\boxed{x(t) = -\frac{1}{\frac{a^{3/2}}{6}\sqrt{\left[1-\left(t/\sqrt{a}\right)^2\right]^3} + C}}$$

(b) $x y^2 \frac{dy}{dx} = 1 + y^3$

The above equation can be written as

$$\frac{y^2}{(1+y^3)} \frac{dy}{dx} = \frac{1}{x} \quad \text{(dividing by } x(1+y^3)\text{)}$$

integrating $\Rightarrow \underbrace{\int \frac{y^2}{(1+y^3)} dy}_{I_1} = \underbrace{\int \frac{dx}{x}}_{I_2}$

$I_2 = \ln|x| + C_1$

$I_1 = \int \frac{y^2 dy}{1+y^3}$

by substitution: $1+y^3 = u$

$\frac{du}{dy} = 3y^2$

$\Rightarrow I_1 = \frac{1}{3} \int \frac{du}{u} = \frac{1}{3} \ln|U| + C_1 = \frac{1}{3} \ln|1+y^3| + C_1$

$\frac{1}{3} \ln|1+y^3| + C_1 = \ln|x| + C_2 \Rightarrow \ln|1+y^3| + 3C_1 = 3\ln|x| + 3C_2$

$\ln|1+y^3| = \overbrace{3\ln|x| + C}^{\ln|x^3| + C}$

$1 + y^3 = e^{\ln|x^3| + C}$

$= x^3 \cdot \underbrace{e^C}_{A}$

$\Rightarrow y^3 = Ax^3 - 1 \Rightarrow \boxed{y = [Ax^3 - 1]^{1/3}}$

(c) $y \cos^2 x \dfrac{dy}{dx} - \tan x - 2 = 0$ (*)

Dividing by $\cos^2 x$, (*) reads $y\dfrac{dy}{dx} - \dfrac{\tan x}{\cos^2 x} - \dfrac{2}{\cos^2 x} = 0$

$$\Rightarrow y\frac{dy}{dx} = \frac{2}{\cos^2 x} + \frac{\tan x}{\cos^2 x}$$

$$\underbrace{\int y\,dy}_{I_1} = \underbrace{\int \left[\frac{2}{\cos^2 x} + \frac{\tan x}{\cos^2 x}\,dx\right]}_{I_2}$$

$I_1 = \dfrac{y^2}{2} + C_1$ (power Rule)

$I_2 = \displaystyle\int \frac{2}{\cos^2 x}\,dx + \int \frac{\tan x\,dx}{\cos^2 x}$

By substitution: if $u = \tan x$

$\dfrac{du}{dx} = \dfrac{1}{\cos^2 x} \Rightarrow du = \dfrac{dx}{\cos^2 x}$

$$\Rightarrow I_2 = \int 2\,du + \int u\,du = 2u + \frac{u^2}{2} + C_2 = 2\tan x + \frac{\tan^2 x}{2} + C_2$$

$$\frac{y^2}{2} + C_1 = 2\tan x + \frac{\tan^2 x}{2} + C_2 \Rightarrow y^2 = 4\tan x + \tan^2 x + C$$

$$\boxed{y = \pm\sqrt{4\tan x + \tan^2 x + C}}$$

(the restriction upon x has been made in order to guarantee that tan x is "well-behaved"; i.e., continuous, non-singular, etc.)

2. $\frac{dy}{dx} = \underbrace{\frac{5xy}{(x+3)(2x+1)}}$ (Initial condition $y(0) = 3$)
1st step: find the general solution

$F(x)G(y) \Rightarrow$ we can apply separation of variables

$$\frac{1}{y}\frac{dy}{dx} = \frac{5x}{(x+3)(2x+1)} \Rightarrow \underbrace{\int \frac{dy}{y}}_{\ln|y|+C_1} = \underbrace{\int \frac{5x\,dx}{(x+3)(2x+1)}}_{I_2}$$

$$I_2 = \int \left(\frac{3}{x+3} - \frac{1}{2x+1} \right) dx$$ (we have used

$$\frac{5x}{(x+3)(2x+1)} = \frac{A}{x+3} + \frac{B}{2x+1} \Rightarrow \frac{A(2x+1) + B(x+3)}{(x+3)(2x+1)} \quad \begin{cases} 2A + B = 5 \\ A + 3B = 0 \end{cases}$$

$\Rightarrow A = 3$
$B = -1$)

$$I_2 = \int \frac{3}{x+3}dx - \int \frac{dx}{(2x+1)} = 3\ln|x+3| - \frac{1}{2}\ln|2x+1| + C_2 =$$

$$= \ln\left|\frac{(x+3)^3}{\sqrt{2x+1}}\right| + C_2$$

$$\Rightarrow \ln|y| = \ln\left|\frac{(x+3)^3}{\sqrt{2x+1}}\right| + \underbrace{\tilde{C}}_{\ln C} \Rightarrow \ln|y| = \ln\left|\frac{C(x+3)^3}{\sqrt{2x+1}}\right| \Rightarrow$$

$$y = \frac{C(x+3)^3}{\sqrt{2x+1}} \Rightarrow y(0) = \frac{C(0+3)^3}{\sqrt{2\cdot 0+1}} \Rightarrow 27C = 3$$

$$\boxed{C = 1/9}$$

$$\boxed{y(x) = \frac{1}{9}\frac{(x+3)^3}{\sqrt{2x+1}}}$$

3. (a) $5\frac{dy}{dt} + 10y = 2 \Rightarrow$ Dividing by 5: $\frac{dy}{dt} + 2y = \frac{2}{5}$

$\Rightarrow$ 1st order inhomogeneous linear differential equation of constant coefficients

- Particular solution: $y_p = C$ (const.)

$\Rightarrow 2 \cdot C = \frac{2}{5} \Rightarrow C = \frac{1}{5} \Rightarrow y_p = \frac{1}{5}$

- Complementary solution:

Associated homogeneous diff. equation:

$\frac{dz}{dt} + 2z = 0 \quad (z = y - \frac{2}{5})$

Solution: $z(t) = Ae^{-2t}$

$\boxed{y(t) = \frac{1}{5} + Ae^{-2t}} \qquad \lim_{t \to \infty} y(t) = \frac{1}{5}$

(b) Solved analogously

$5\frac{dy}{dt} + 10y = 2 \quad \Rightarrow \frac{dy}{dt} - 2y = \frac{2}{5}$

- Particular solution: $y_p = C \Rightarrow \cancel{y_p} - 2 \cdot C = \frac{2}{5} \Rightarrow C = -\frac{1}{5}$; $y_p = \frac{-1}{5}$

- Complementary solution $z(t) = Ae^{+2t}$

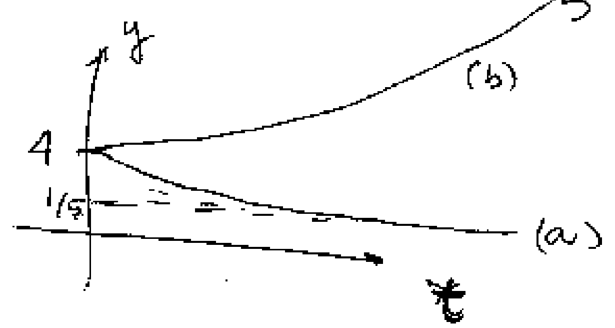
General solution: $\boxed{y(t) = -\frac{1}{5} + Ae^{2t}}$ $\lim_{t \to \infty} y(t) = \infty$

- Differences: The solution in (a) decays exponentially towards a stationary value, whereas the solution in (b) increases exponentially. In (a), the stationary value is its particular solution

- $y(0) = 4$

$(a) \Rightarrow \frac{1}{5} + A = 4 \Rightarrow A = 4 - \frac{1}{5} = \frac{19}{5} = 3{,}8$

$(b) = -\frac{1}{5} + A = 4 \Rightarrow A = 4 + \frac{1}{5} = \frac{21}{5} = 4.2$



4.

$$\frac{dy}{dx} - 3y = 6x - 3$$

- Step 1: General solution: particular solution + solution of the associated homogeneous equation

Particular solution: "Guess": $y_p = ax + b$

$$\frac{dy_p}{dx} = a \Rightarrow \frac{dy_p}{dx} - 3y_p = -3ax + (a - 3b) = 6x - 3$$

$$\Rightarrow 6 = -3a \Rightarrow a = -2 \quad ; \quad a - 3b = -3$$

$$-2 - 3b = -3$$

$$b = -1/3$$

$$y_p = -2x - 1/3$$

Complementary solution: satisfies $\frac{dz}{dx} - 3z = 0$

with $z = y - y_p$

$$z = Ae^{3x} \Rightarrow y(x) = Ae^{3x} - 2x - 1/3$$

- Step 2: $y(0) = 1 \Rightarrow A - \frac{1}{3} = 1 \Rightarrow A = \frac{2}{3}$ $\boxed{y(x) = \frac{2}{3}e^{3x} - \frac{1}{3} - 2x}$