Further mathematics for economists - Exercise sheet 2 - solutions

1. (a) $I = \int \frac{dx}{x^2+5x+6}$

(*) Hint: reduce the integrand to the form $\frac{A}{x+a} + \frac{B}{x+b}$

• 1st step: note that $x^2+5x+6 = (x+3)(x+2)$

Hence $I = \int \frac{dx}{(x+3)(x+2)}$

Using (*): $\frac{1}{(x+3)(x+2)} = \frac{A}{(x+3)} + \frac{B}{(x+2)} = \frac{A(x+2)+B(x+3)}{(x+3)(x+2)} =$

$= \frac{(A+B)x + 2A+3B}{(x+3)(x+2)}$

The numerators must be equal:

LHS : 1

RHS : $(A+B)x + 2A+3B$ (1st-degree polynomial)

The LHS can be written as $0 \cdot x + 1$

Imposing that ~~Setting~~ the terms with equal powers be equal we have:

$A+B=0 \Rightarrow A = -B$

$2A+3B=1 \Rightarrow -2B+3B=1 \Rightarrow \boxed{B=1}$ and thus $\boxed{A=-1}$

Thus $I = \underbrace{-\int \frac{dx}{x+3}}_{I_1} + \underbrace{\int \frac{dx}{x+2}}_{I_2}$

I_1 : by substitution: $u = x+3 \Rightarrow du = dx$

$I_1 = \int \frac{du}{u} = \ln u + C_1 = \ln(x+3) + C_1$, $x+3>0$

I_2 : by substitution: $u = x+2 \Rightarrow du = dx$

$I_2 = \int \frac{du}{u} = \ln u + C_2 = \ln(x+2) + C_2$, $x+2>0$

$\boxed{I = -\ln(x+3) - C_1 + \ln(x+2) + C_2 = \ln(x+2) - \ln(x+3) + C}$

and $I = \ln|x+3| + \ln|x+2| + \underbrace{c_1 + c_2}_{C}$

(b) $I = \int \frac{dx}{1+e^x}$

• Hint 1: Integration by substitution: $e^x = u \Rightarrow \frac{du}{dx} = e^x$

$$\frac{du}{\underbrace{e^x}_{u}} = dx$$

Then $I = \int \frac{du}{u(u+1)}$

• Hint 2: the same trick as in (a)

$$\frac{1}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1} = \frac{Au + A + Bu}{u(u+1)}$$

$A + B = 0 \Rightarrow A = -B$
$A = 1$
$B = -1$

$$\Rightarrow I = +\int \frac{du}{u} - \int \frac{du}{(u+1)} = +\ln(u) - \ln(u+1) + C$$

$$\Rightarrow \boxed{I = -\ln(e^x) + \ln(e^x+1) + C = +x - \ln(e^x+1) + C}$$

(c) $\int \ln x \, dx$

Integration by parts: $f(x) = \ln x \to f'(x) = \frac{1}{x}$

$g'(x) = 1 \to g(x) = \int dx = x$

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

$$\int \ln x \, dx = x \ln x - \int \frac{1}{x} \cdot x \, dx = x \ln x - x$$

(d) $\int x^2 \cos x \, dx$

Integration by parts:

$x^2 = f(x)$ $\qquad$ $f'(x) = 2x$

$\cos x = g'(x) \Rightarrow$ $\qquad$ $g(x) = \int \cos x \, dx = \sin x$

$$\Rightarrow \int x^2 \cos x \, dx = x^2 \sin x - \underbrace{\int 2x \sin x \, dx}_{(*)}$$

(*) will be integrated by parts again

$$\int 2x \sin x \, dx = 2 \int \underset{f(x)}{\underset{\downarrow}{x}} \, \underset{g'(x)}{\underset{\downarrow}{\sin x}} \, dx$$

Hence $f'(x) = 1$
$g(x) = -\cos x$

and $(*) = 2 \left[-x \cos x + \underbrace{\int \cos x \, dx}_{\sin x} \right]$

$$\Rightarrow \boxed{\int x^2 \cos x \, dx = x^2 \sin x - \left[-2x \cos x + 2 \sin x \right] = 2x \cos x + (x^2 - 2) \sin x}$$

(e) $\int \frac{dx}{a+x^2}$, a constant.

• First step : $I = \int \frac{dx}{a+x^2} = \frac{1}{a}\int \frac{dx}{(1+\frac{x^2}{a})}$

* integral by substitution : 1st change of variable :

$u = \frac{x}{\sqrt{a}}$

$\frac{du}{dx} = \frac{1}{\sqrt{a}}$

$\Rightarrow dx = \sqrt{a}\, du$

$\Rightarrow I = \frac{1}{a}\int \frac{\sqrt{a}\, du}{(1+u^2)} = \frac{1}{\sqrt{a}}\int \frac{du}{(1+u^2)}$

* Integral by substitution : 2nd change of variable

$u = \tan v \Rightarrow 1+u^2 = 1+\tan^2 v = \sec^2 v$

$\frac{du}{dv} = \sec^2 v$

$du = \sec^2 v\, dv$

$I = \frac{1}{\sqrt{a}}\int \frac{\cancel{\sec^2 v}\, dv}{\cancel{\sec^2 v}} = \frac{1}{\sqrt{a}} v + C$

But $v = \text{Arc}\tan u = \text{Arc}\tan\left(\frac{x}{\sqrt{a}}\right)$

$$\boxed{I = \frac{1}{\sqrt{a}}\text{Arc}\tan\left(\frac{x}{\sqrt{a}}\right) + C}$$

(f) $\int \sqrt{1-x^2}\,dx$

Hint 1 $\Rightarrow \int \sqrt{1-x^2}\,dx = \int \frac{1-x^2}{\sqrt{1-x^2}}\,dx =$

$$= \underbrace{\int \frac{dx}{\sqrt{1-x^2}}}_{I_1} - \underbrace{\int \frac{x^2\,dx}{\sqrt{1-x^2}}}_{I_2}$$

$I_1 = \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C_1$ (solved last lecture)

if you don't remember: use integration by substitution

$I_2 = \int \frac{x^2\,dx}{\sqrt{1-x^2}}$ (will be integrated by parts)

Trick: $\int \frac{x^2\,dx}{\sqrt{1-x^2}} = \int \frac{x \cdot x\,dx}{\sqrt{1-x^2}}$

$f(x) = x \Rightarrow f'(x) = 1$

$g'(x) = \frac{x}{\sqrt{1-x^2}} \Rightarrow g(x) = \int \frac{x\,dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$

(also solved last lecture)

if you don't remember: use integration by substitution

Hence $I_2 = f(x)\,g(x) - \int f'(x)\,g(x)\,dx$

$\Rightarrow I_2 = -x\sqrt{1-x^2} + \int \sqrt{1-x^2}\,dx$

$\Rightarrow \int \sqrt{1-x^2}\,dx = \arcsin x + C_1 - \left[-x\sqrt{1-x^2} + \int \sqrt{1-x^2}\,dx\right]$

This looks bad!!! I came back to the integral I had at the VERY beginning

$$\int \sqrt{1-x^2}\,dx = \arcsin x + C_1 + x\sqrt{1-x^2} - \int \sqrt{1-x^2}\,dx \qquad (*)$$

However, note that, if I call $I = \int \sqrt{1-x^2}\,dx$, I can write (*) as

$$I = \arcsin x + C_1 + x\sqrt{1-x^2} - I \Rightarrow$$

$$\Rightarrow 2I = \arcsin x + x\sqrt{1-x^2}$$

$$\Rightarrow I = \frac{1}{2}\left[\arcsin x + x\sqrt{1-x^2} + C_1\right]$$

$$\boxed{\Rightarrow \int \sqrt{1-x^2}\,dx = \frac{1}{2}\arcsin x + \frac{x}{2}\sqrt{1-x^2} + \overset{C}{\frac{C_1}{2}}}$$

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