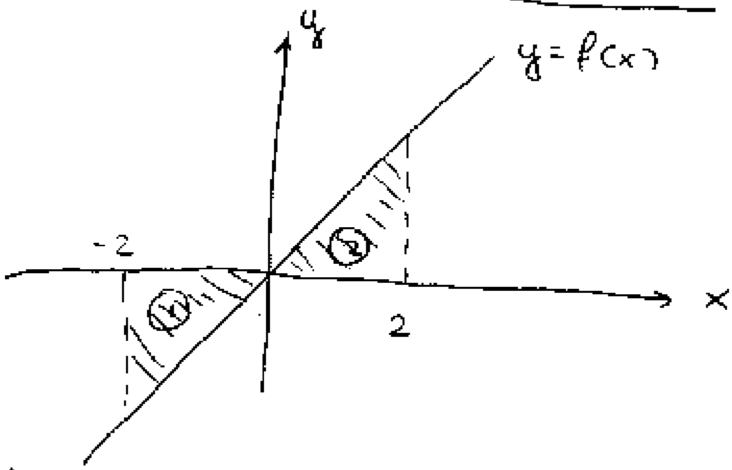
# Further mathematics for economists - Exercise sheet 1 - Solutions

**1.** (a) $\int_{-2}^{2} x\,dx = \frac{x^2}{2}\Big|_{-2}^{2} = 0$ (*) Sketch of the function:




- Area to be calculated: area ① + area ②

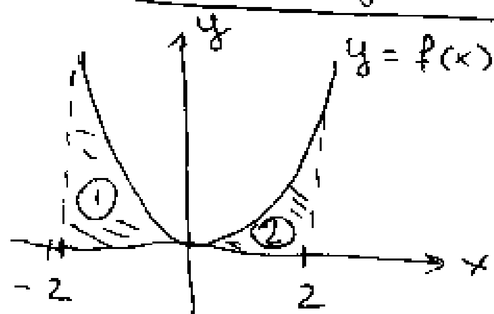
Please note: if the function to be integrated lies below the x axis, its integral is by convention negative

- Area ①: $\int_{-2}^{0} x\,dx = \frac{0}{2} - \frac{4}{2} = -2$

- Area ②: $\int_{0}^{2} x\,dx = 2$

$\Rightarrow$ area ① + area ② = 0

(b) $\int_{-2}^{2} 3x^2 dx = x^3\Big|_{-2}^{2} = 16$ (*) Sketch of the function:




- Area to be calculated: area ① + area ② (both integrals > 0)

- Area ①: $\int_{-2}^{0} 3x^2 dx = x^3\Big|_{-2}^{0} = 0^3 - (-2)^3 = 8$

- Area ②: $\int_{0}^{2} 3x^2 dx = x^3\Big|_{0}^{2} = 8$

ase note: with this property one can "guess" integrals of more complicated functions

Ple

Examples $\displaystyle\int_{-1}^{1} x^3 \cos x \, dx = 0$ • note that $x^3 \cos x$ is an ODD function

$f(x) = x^3 \cos x$

$f(-x) = (-x)^3 \cos(-x) = -x^3 \cos x = -f(x)$

$\displaystyle\int_{-0.5}^{0.5} \frac{x\,dx}{1+x^2} = 0$ • again, $\frac{x}{1+x^2}$ is an odd function

$f(x) = \frac{x}{1+x^2}$

$f(-x) = \frac{-x}{1+x^2} = -f(x)$

- Difference between (a) and (b):

(a): the function to be integrated is an odd function

(ODD function: $f(x) = -f(-x)$)

$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 0 \quad \text{in this case}$$

(b): the function to be integrated is an even function

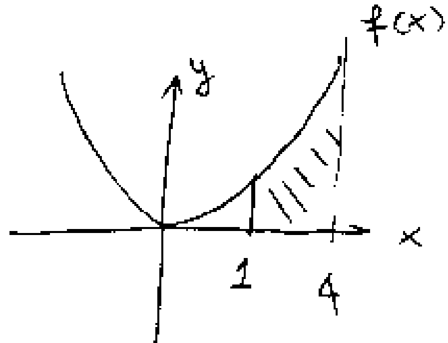
(EVEN function: $f(x) = f(-x)$)

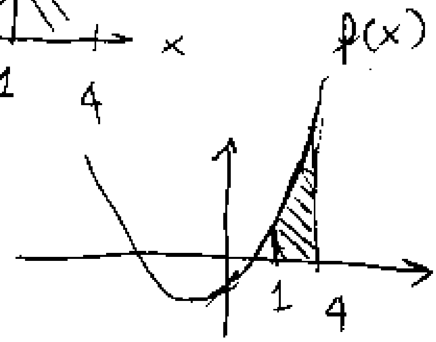
$$\Rightarrow \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \quad \text{in this case}$$

2.

(a) $y(x) = x^2 \Rightarrow$

$$A = \int_1^4 x^2\,dx = \frac{x^3}{3}\Big|_1^4 =$$



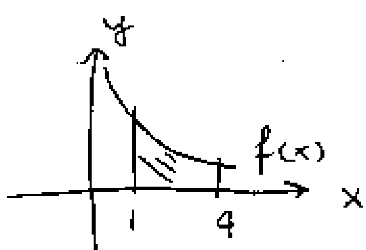

(b) $y(x) = 2x + x^2 - 1$

$$A = \int_1^4 2x + x^2 - 1 = x^2 + \frac{x^3}{3} - x\Big]_1^4 = (4)^2 + \frac{(4)^3}{3} - 4 - \left[(1) + \frac{(1)^3}{3} - 1\right] = 33$$

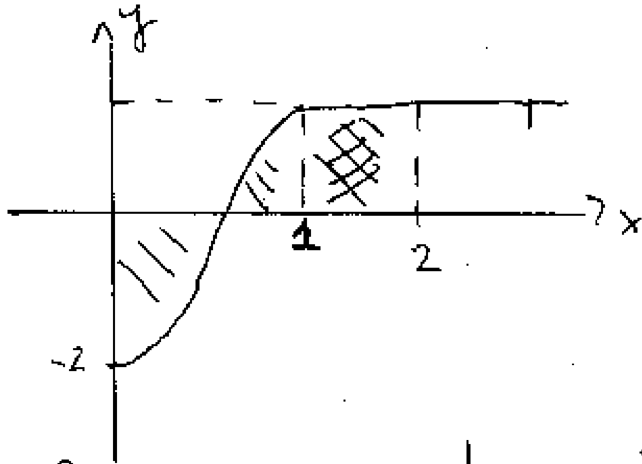
(c) $y(x) = \frac{1}{x^2}$

$$A = \int_1^4 \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_1^4$$

$$= -\frac{1}{4} - \left(-1\right) = \frac{3}{4}$$



3. $f(x) = \begin{cases} 4x^3 - 2, & x < 1 \\ 2, & x \geq 1 \end{cases}$

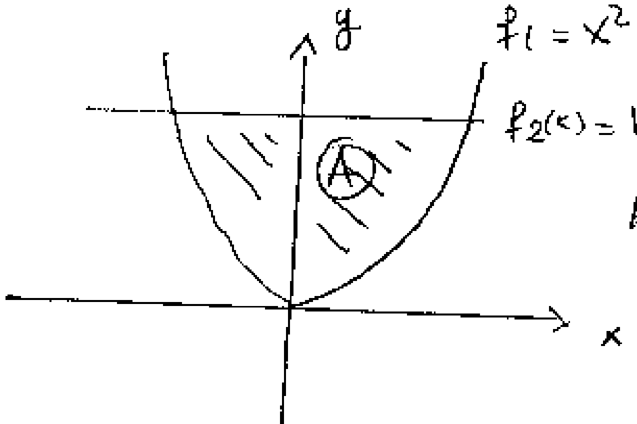



$$I = \int_0^2 f(x) = \int_0^1 (4x^3 - 2)\,dx + \int_1^2 2\,dx = (x^4 - 2x)\Big|_0^1 + (2x)\Big|_1^2 =$$

$$= \underbrace{(1)^4 - 2\cdot 1}_{-1} - 0 + \underbrace{2\cdot 2 - 2\cdot 1}_{2} = 1$$

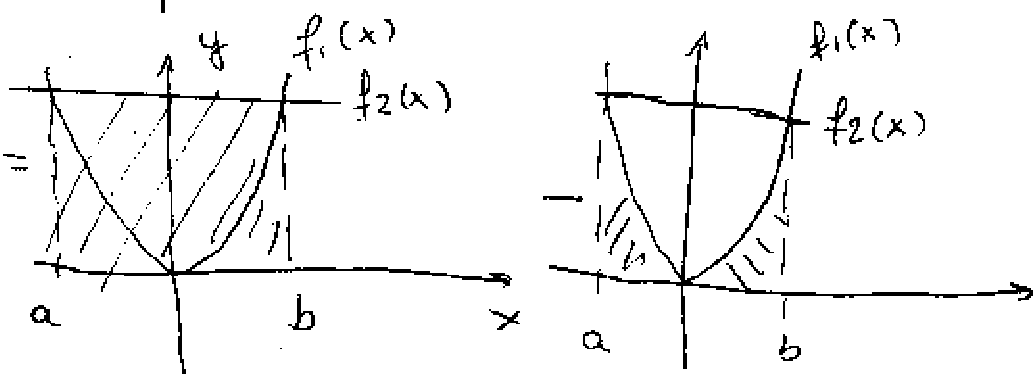
(*) Please note: In this case, f(x) is given by different expressions in different ranges of x. In this case we split the integral as in the example above.

4.




How to find A ?

A =




( Area below the line − area below the parabola, in the interval $x \in [a, b]$ )

• First step: find $a, b$

⇒ Such points are the intersection of the 2 curves. Hence, they must satisfy $f_1(x) = f_2(x)$

$$x^2 = k \Rightarrow x = \pm\sqrt{k}$$

$A_1$: Area bellow the line : $\displaystyle\int_{-\sqrt{k}}^{\sqrt{k}} f_2(x)\,dx = \int_{-\sqrt{k}}^{+\sqrt{k}} k\,dx = kx\Big|_{-\sqrt{k}}^{+\sqrt{k}} = k^{3/2} - (-k^{3/2})$

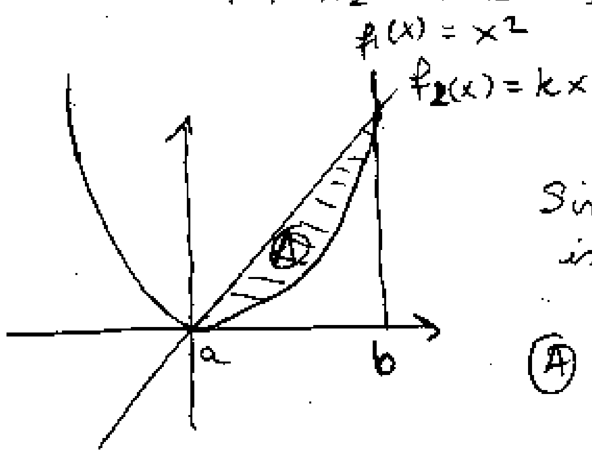
$= 2k^{3/2}$

$A_2$: Area below the parabola: $\displaystyle\int_{-\sqrt{k}}^{\sqrt{k}} x^2 dx = \frac{x^3}{3}\Big|_{-\sqrt{k}}^{\sqrt{k}} = \frac{(\sqrt{k})^3}{3} - \frac{(-\sqrt{k})^3}{3} =$

$= \dfrac{2}{3}k^{3/2}$

$$A = A_1 - A_2 = 2k^{3/2} - \frac{2}{3}k^{3/2} = \frac{4}{3}k^{3/2}$$

**5.**




Similar to 4 (only one of the functions is different)

(A) → area bellow the line − area below the parabola

in the range $[a, b]$.

• Find $a, b$: For such points, $f_1(x) = f_2(x) \Rightarrow x^2 = kx$ or

$$x^2 - kx = 0 \quad \begin{cases} x_1 = 0 \\ x_2 = k \end{cases}$$

Note that, with respect to 4, the range of integration was changed

$\rightarrow A_1 = \displaystyle\int_0^k kx\,dx = k\frac{x^2}{2}\Big|_a^k = \frac{k^3}{2}$; $A_2 = \displaystyle\int_a^k x^2 dx = \frac{x^3}{3}\Big|_a^k = \frac{k^3}{3}$

$$A = \frac{k^3}{2} - \frac{k^3}{3} = \frac{3k^3 - 2k^3}{6} = \frac{k^3}{6}$$

6. (a) $\int \frac{x^2+1}{x}\,dx = \int x\,dx + \int \frac{1}{x}\,dx = \frac{x^2}{2} + C_1 + \ln X + C_2 =$

$= \frac{x^2}{2} + \ln x + C \quad \underbrace{, x > 0}_{(*)} \quad$ , or $\frac{x^2}{2} + \ln|x| + C$

(*) Please note: we always write the condition (*) because $\ln x$ is only defined for $x > 0$

However, for $x < 0$, $\ln(-x)$ is defined

($-x$ is positive in this case)

$$\frac{d}{dx}\ln(-x) = \frac{d\ln(-x)}{d(-x)}\,\frac{d(-x)}{dx} = -1 \times \frac{1}{(-x)} = \frac{1}{x}$$

$\Rightarrow$ one can put both cases together if we write $\frac{d}{dx}\ln|x| = \frac{1}{x}$

$\Rightarrow \int \frac{1}{x}\,dx = \ln|x|$

(b) $\int e^{ax}\,dx$

We will apply the following trick

~~[illegible]~~ Let's use it. $u = ax$

$$\frac{du}{dx} = a \Rightarrow \frac{du}{dx} \times dx = a\,dx$$

$$\frac{du}{a} = dx$$

$$\int e^{ax}\,dx = \int e^u \frac{du}{a} = \frac{1}{a}\left[e^u + C\right] =$$

But, since $u = ax$, we may write

$$\int e^{ax} dx = \frac{e^{ax}}{a} + \frac{c}{a}$$

$= c$

(c) $\int 2x e^{x^2} dx$

Similarly to (b) we take $u = x^2$

Then $\frac{du}{dx} = 2x$ and $\frac{du}{\cancel{dx}} \cdot \cancel{dx} = 2x\,dx$

$$\Rightarrow \int 2x e^{x^2} dx = \int e^u du = e^u + C = e^{x^2} + C$$

(d) $$\int \left( \frac{6}{x^5} + \frac{1}{x^3} + \frac{2}{\sqrt{x^3}} \right) dx = -\frac{6}{4 \times 4} - \frac{1}{2x^2} - \frac{2 \times 2}{\sqrt{x}} + C$$

$$= \frac{-3}{2x^4} - \frac{1}{2x^2} - \frac{4}{\sqrt{x}} + C$$