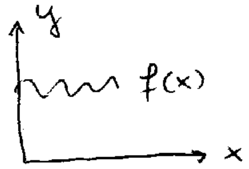
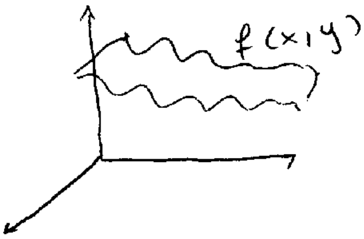
FUNCTIONS OF SEVERAL VARIABLES

I- Introduction

(*) Function of one variable: $f(x)$ (represented as a curve in a plane)



(*) Function of two variables: $f(x,y)$ (represented as a surface in the 3D space)



(*) Function of n variables: $f(x_1, \dots, x_n)$ (represented as a hypersurface in the ~~n+1~~ n-dimensional space)

II. Differentiation

(*) Problem: analyze how $f(x,y)$ changes when x and y change

1. Partial differentiation: one differentiates with respect to one variable, while the other variable is kept fixed

- Partial differentiation with respect to x: $\dfrac{\partial f(x,y)}{\partial x} = \lim\limits_{h \to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$
- Partial differentiation with respect to y: $\dfrac{\partial f(x,y)}{\partial y} = \lim\limits_{k \to 0} \dfrac{f(x,y+k) - f(x,y)}{k}$

(*) Please note: the methods and rules of partial differentiation are those of ordinary differentiation

Example: Find the first and second derivatives of $f(x,y) = x^2 e^y$

- First derivatives

$$\frac{\partial f}{\partial x} = 2x e^y$$
$$\frac{\partial f}{\partial y} = x^2 e^y$$

- Second derivatives:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(2x e^y\right) = 2e^y$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(x^2 e^y\right) = x^2 e^y$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 2x e^y \; ; \; \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 2x e^y$$

Please note: $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are called "mixed derivatives"

- For a smooth function (the 1^{st}, 2^{nd} derivatives exist and are continuous), $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$

1.1 - Gradient:

1.1.1 - Definition: The gradient is the vector whose components are the first partial derivatives of f

Two variables

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix}$$

or

Df

(∇ = "nabla")

n variables

$$\nabla f \text{ or } Df = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

Example: Write the gradient of $f(x,y) = x^2 e^y$

$$Df = \begin{pmatrix} 2xe^y \\ x^2 e^y \end{pmatrix}$$

The gradient at a point (x,y) defines a plane tangent to the surface $f(x,y)$ at such a point

1.2 - Hessian matrix

1.2.1 - Definition: the Hessian matrix is the matrix composed by the second partial derivatives of f

Two variables

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

n variables

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & & & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Example: Write the Hessian matrix of $f(x,y) = x^2 e^y$

$$D^2 f = \begin{pmatrix} 2e^y & 2xe^y \\ 2xe^y & x^2 e^y \end{pmatrix}$$

Please note: For smooth functions the Hessian matrix is symmetric

2 - The chain rule

(Generalization of the composite rule for a function of 2 or more variables).

2.1 - Small increments

Let us consider $z = f(x,y)$ and the small increments h and k in x and y, respectively. Then $f(x+h, y+k) \approx h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} = \Delta z = \Delta f$

Proof: increment in x: $h\frac{\partial f}{\partial x} \approx f(x+h,y) - f(x,y)$ $(*)$

Subsequently: increment in y: $k\frac{\partial f}{\partial y} = f(x+h,y+k) - f(x+h,y)$ $(**)$

$(*)+(**) = f(x+h,y+k) - \cancel{f(x+h,y)} + \cancel{f(x+h,y)} - f(x,y)$

Please note: $\lim_{\Delta z \to 0} = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = dz$ (this is called "the differential")

2.2 - The chain rule

Let $t=f(x,y)$, where f is smooth and $x = x(t)$, $y = y(t)$

Then $z = f(t) = f(x(t), y(t))$ and $\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$

Proof: • Divide the small increments formula by Δt:

$$\frac{\Delta f}{\Delta t} = \frac{\partial f}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial f}{\partial y}\frac{\Delta y}{\Delta t}$$

• Take $\lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

Example:

find $\frac{df}{dt}$ with $f(x,y) = x^2 y$ and $y = 5t$, $x = 2t$

Chain rule: $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$\frac{\partial f}{\partial x} = 2xy$, $\frac{\partial f}{\partial y} = x^2$, $\frac{dy}{dt} = 5$, $\frac{dx}{dt} = 2$

$$\frac{df}{dt} = 2xy \cdot 2 + 5 \cdot x^2 = \underbrace{4(5t)(2t)}_{40t^2} + \underbrace{5 \times 4t^2}_{20} = 60t^2$$

2.2.1 - Particular case (total derivative)

If $t = x$ then $\boxed{\frac{df}{dx} = \frac{\partial f}{\partial x}\underbrace{\frac{dx}{dx}}_{1} + \frac{\partial f}{\partial y}\frac{dy}{dx}}$

⇒ This is a total derivative

Example: Find $\frac{df}{dx}$ with $f(x,y) = x^2 y$ and $y = 2x$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 2xy + x^2 \cdot 2 = 4x^2 + 2x^2 = 6x^2$$

(*) Please note: The chain rule holds for n independent variables.

- If $f = f(x_1, \dots, x_n)$ and $x_1 = x_1(t), x_2 = x_2(t) \dots, x_n = x_n(t)$

then $\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \dots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$

- More compact notation:

$$\frac{df}{dt} = [Df]^T \cdot \frac{d\vec{x}}{dt}, \text{ with } \vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ and } Df = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}$$

3. Homogeneous functions

3.1 - Definition: Let $f(x_1, \dots, x_n)$ be defined for all positive $x_1, \dots, x_n$ and let r be a real number. The function f is homogeneous of degree ~~r~~ if

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^r f(x_1, \dots, x_n)$$

- In general: λ can take any value
- For simplicity (and to avoid trouble with fractional exponents) $\lambda > 0$.

- Example 1: The function $f(x,y,w) = \frac{x}{y} + \frac{2w}{3x}$ is homogeneous, with $r = 0$ ("homogeneous of degree 0")

Check:

$$f(\lambda x, \lambda y, \lambda w) = \frac{\cancel{\lambda} x}{\cancel{\lambda} y} + \frac{2\cancel{\lambda} w}{3\cancel{\lambda} x} = \frac{x}{y} + \frac{2w}{3x} = f(x,y,w)$$
$$= \lambda^0 f(x,y,w)$$

Example 2: The function $f(x,y,z) = ax^3 + by^3 + cxyz + dz^3$ is homogeneous with $r=3$ (homogeneous of degree 3)

$$f(\lambda x, \lambda y, \lambda z) = a\lambda^3 x^3 + b\lambda^3 y^3 + c\lambda^3 xyz + d\lambda^3 z^3$$
$$= \lambda^3 (ax^3 + by^3 + cxyz + dz^3)$$
$$= \lambda^3 f(x,y,z)$$

2 - Euler's theorem

The function $f(x_1, \dots, x_n)$ is homogeneous of degree r ~~iif~~ if and only if

$$x_1 \frac{\partial f}{\partial x_1} + \dots + x_n \frac{\partial f}{\partial x_n} = r f(x_1, \dots, x_n)$$

Proof: Let us consider $F(s,x,y) = s^{-r} f(sx, sy)$

f homogeneous: $f(sx, sy) = s^r f(x,y)$

Then $F(s,x,y) = s^{-r} s^r f(x,y) = f(x,y)$

(Does not depend on s : $\frac{dF}{ds} = 0$)

$$\frac{dF}{ds} = s^{-r} \frac{\partial f(sx,sy)}{\partial (sx)} \overbrace{\frac{d(sx)}{ds}}^{x} + s^{-r} \frac{\partial f(sx,sy)}{\partial (sy)} \overbrace{\frac{d(sy)}{ds}}^{y} - r s^{-r-1} f(sx,sy)$$

If $F(s,x,y)$ does not depend on s then $F(s,x,y) \equiv F(1,x,y)$ and the above-stated relation also holds for $s=1$

Hence,

$$\boxed{0 = x \frac{\partial f(x,y)}{\partial x} + y \frac{\partial f(x,y)}{\partial y} - r f(x,y)}$$

* Please note : the chain rule holds for n independent variables.

If $f = f(x_1, \dots, x_n)$ and $x_1 = x_1(t)$, $\vdots$, $x_n = x_n(t)$ then $\frac{df}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\frac{dx_2}{dt} + \dots + \frac{\partial f}{\partial x_n}\frac{dx_n}{dt}$

3. Implicit functions

3.1 - Definition :

A function $y = f(x)$ is implicit if it is written as the equation $F(x,y) = 0$

Example : $x^2+y^2 = 1$ (eq. of a circle of radius 1) is an implicit function.

Explicitly, $y = \pm\sqrt{1-x^2}$ (+ upper half circle, − lower half circle) and $\frac{dy}{dx} = \frac{\pm(-x)}{\sqrt{1-x^2}} = \frac{\mp x}{\sqrt{1-x^2}}$

3.2 - Method for differentiating implicit functions

1) Write $\frac{dF}{dx} = \frac{\partial F}{\partial y}\frac{dy}{dx} + \frac{\partial F}{\partial x}\frac{dx}{dx} = 0 \quad (*)$

2) Rearrange $(*)$: $\boxed{\frac{dy}{dx} = \frac{-\partial F/\partial x}{\partial F/\partial y} \quad \left(\frac{\partial F}{\partial y} \neq 0\right)}$

Example : find $\frac{dy}{dx}$ with $x^2+y^2 = 1$

In this case $F(x,y) = x^2 + y^2 - 1 = 0$

$0 = \frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} \Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{x}{y} = \frac{\mp x}{\sqrt{1-x^2}}$

* Please note : In case $F = F(x_1, \dots, x_n) = 0$, we can differentiate this function in the same way

3.3 - Jacobian matrices

Let us consider the implicit relations $F_1(x,y,z) = 0$ and $F_2(x,y,z) = 0$ with $y = y(x)$ and $z = z(x)$. Applying 3.2 we have

$$\frac{dF_1}{dx} = \frac{\partial F_1}{\partial x}\overbrace{\frac{dx}{dx}}^{1} + \frac{\partial F_1}{\partial y}\frac{dy}{dx} + \frac{\partial F_1}{\partial z}\frac{dz}{dx} = 0 \quad (*)$$

$$\frac{dF_2}{dx} = \frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial y}\frac{dy}{dx} + \frac{\partial F_2}{\partial z}\frac{dz}{dx} = 0 \quad (**)$$

(*) and (**) may be written as

$$\underbrace{\begin{bmatrix} \partial F_1/\partial y & \partial F_1/\partial z \\ \partial F_2/\partial y & \partial F_2/\partial z \end{bmatrix}} \begin{bmatrix} \frac{dy}{dx} \\ dz/dx \end{bmatrix} = - \begin{bmatrix} \partial F_1/\partial x \\ \partial F_2/\partial x \end{bmatrix}$$

$J \equiv$ Jacobian matrix of F_1, F_2 with respect to the variables y, z.

so that $\begin{bmatrix} dy/dx \\ dz/dx \end{bmatrix} = -J^{-1} \begin{bmatrix} \partial F_1/\partial x \\ \partial F_2/\partial x \end{bmatrix}$

(*) Please note: If $\frac{dy}{dx}$, $\frac{dz}{dx}$ exist J is invertible ($\det J \neq 0$)

Example: Given the implicit relations

$$F_1 = y^2 - z^2 - x = 0$$
$$F_2 = yz - x/2 = 0$$

. Find the Jacobian matrix of F_1, F_2 with respect to y, z

$$J = \begin{pmatrix} \partial F_1/\partial y & \partial F_1/\partial z \\ \partial F_2/\partial y & \partial F_2/\partial z \end{pmatrix}$$

$$\frac{\partial F_1}{\partial y} = 2y, \quad \frac{\partial F_1}{\partial z} = -2z, \quad \frac{\partial F_2}{\partial y} = z \quad \frac{\partial F_2}{\partial z} = y$$

$$J = \begin{pmatrix} 2y & -2z \\ z & y \end{pmatrix} \quad (\text{Det } J = 2y^2 + 2z^2)$$

(b). Find $\frac{dy}{dx}$ and $\frac{dz}{dx}$ $\Rightarrow$ $\begin{bmatrix} dy/dx \\ dz/dx \end{bmatrix} = -J^{-1} \begin{bmatrix} \partial F_1/\partial x \\ \partial F_2/\partial x \end{bmatrix}$

$$J^{-1} = \frac{1}{\text{Det } J} \cdot \begin{bmatrix} y & 2z \\ -z & 2y \end{bmatrix} = \frac{1}{(2y^2 + 2z^2)} \begin{bmatrix} y & 2z \\ -z & 2y \end{bmatrix}$$

(Remember: if $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $M^{-1} = \frac{1}{\text{Det } M} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$)

$$\begin{bmatrix} dy/dx \\ dz/dx \end{bmatrix} = -J^{-1} \begin{bmatrix} \partial F_1/\partial x \\ \partial F_2/\partial x \end{bmatrix} = -J^{-1} \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} = \frac{-1}{(2y^2+2z^2)} \begin{bmatrix} y & 2z \\ -z & 2y \end{bmatrix} \begin{bmatrix} -1 \\ -1/2 \end{bmatrix} = \begin{bmatrix} \frac{-y-z}{(2y^2+2z^2)} \\ \frac{z-2y}{(2y^2+2z^2)} \end{bmatrix}$$

Some applications to economics:

- Function of two variables: $Q = F(K, L)$ — Q: production function; K: capital; L: labour

- Partial derivatives:
  - $\frac{\partial F}{\partial K}$ (marginal product of capital)
  - $\frac{\partial F}{\partial L}$ (" " of labor)

Example: $Q = A K^{\alpha} L^{\beta}$ (A, α, β positive constants)

$$\frac{\partial Q}{\partial K} = A \alpha K^{\alpha-1} L^{\beta} = \frac{\alpha}{K} Q$$

$$\frac{\partial Q}{\partial L} = A \beta K^{\alpha} L^{\beta-1} = \frac{\beta}{L} Q$$

- Implicit relations: $F(K, L) = \bar{Q}$ (isoquant) — $\bar{Q}$: a particular, constant output

we can define

$G = F(K, L) - \bar{Q} = 0$

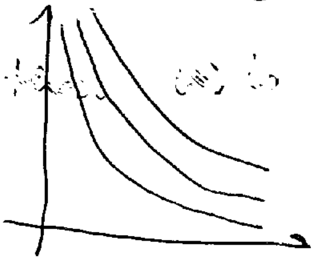
Slope of the isoquant (dL/dK)

$$\frac{dG}{dK} = \frac{\partial F}{\partial K} + \frac{\partial F}{\partial L}\frac{dL}{dK} = 0 \Rightarrow \frac{dL}{dK} = \frac{-\partial F/\partial K}{\partial F/\partial L}$$

$\left|\frac{dL}{dK}\right|$ = marginal of substitution of labor for capital)

Example: $Q = A K^{\alpha} L^{\beta} \Rightarrow \frac{dL}{dK} = \frac{-\alpha/K\, Q}{\beta/L\, Q} = -\frac{\alpha L}{\beta K}$ (*)

- $L(K) = \left(\frac{\bar{Q}}{A K^{\alpha}}\right)^{1/\beta}$



Suppose now I only had (*): $\frac{dL}{dK} = -\frac{\alpha L}{\beta K}$ (separable diff. eq.)

then $\int \frac{dL}{L} = -\int \frac{\alpha}{\beta}\frac{dK}{K}$

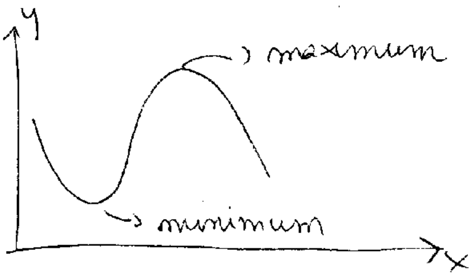
$\ln|L| = -\frac{\alpha}{\beta}\ln|K| + \ln C \Rightarrow L = \left(\frac{1}{K}\right)^{\alpha/\beta} C$

4. Critical points

4.1 - Introduction

• Functions of one variable:
A point (x_0, y_0) is a critical point of $y = f(x)$ if $\left.\frac{df}{dx}\right|_{x=x_0} = 0$

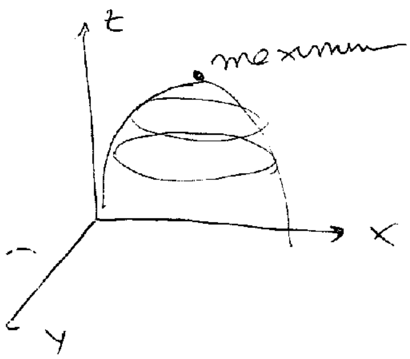
Example:

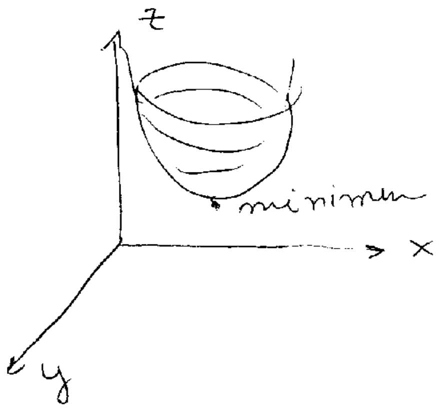


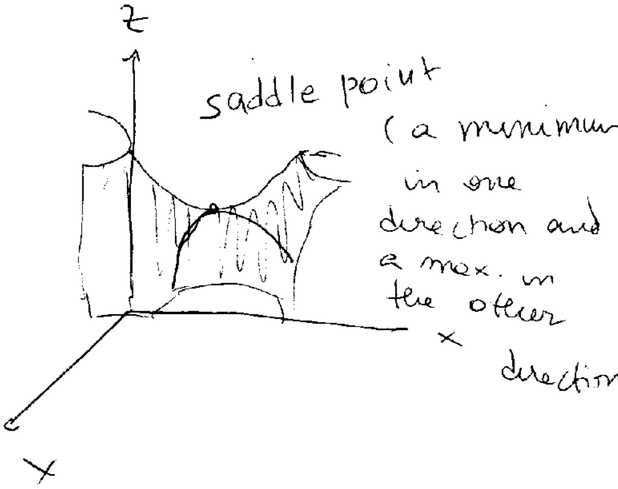
• Functions of more than one variable (e.g. two variables):
A point (x_0, y_0, z_0) is a critical point of $z = f(x,y)$ if

$$\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} = 0 \quad ; \quad \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}} = 0$$

Examples:





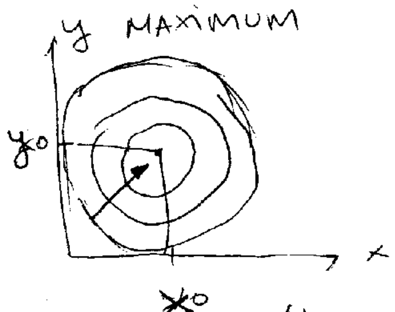


Contour diagrams: allow drawing pictures of critical points of functions of two variables in a plane

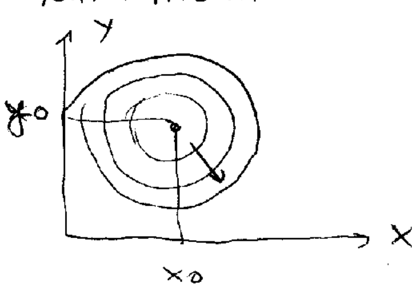
• Contour of a function: $f(x,y) = k$ (k = const.) (one makes a "cut" on $f(x,y)$ parallel to $z = 0$) (the arrows indicate the direction towards which the function grows)

Examples

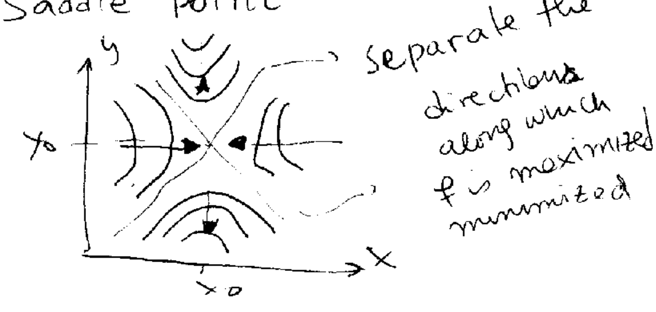
MAXIMUM



MINIMUM



Saddle point



How to determine the nature of critical points?

4.2 - Classification

4.2.1 Functions of one variable : $y = f(x)$:

(*) **Extrema** : (x_0, y_0) so that $\left.\frac{dy}{dx}\right|_{x=x_0} = 0$, or $dy = 0$

- **Maxima** :
  - Sufficient condition : $\left.\frac{d^2y}{dx^2}\right|_{x=x_0} < 0$ or $d^2y < 0$
  - Necessary condition : $\left.\frac{d^2y}{dx^2}\right|_{x=x_0} \leq 0$ or $d^2y \leq 0$
- **Minima** :
  - Sufficient condition : $\left.\frac{d^2y}{dx^2}\right|_{x=x_0} > 0$ or $d^2y > 0$
  - Necessary condition : $\frac{d^2y}{dx^2} \geq 0$ or $d^2y \geq 0$

(*) Please note : these conditions can also be written in terms of the "differential" dy and the "second differential" d^2y

4.2.2 - Functions of several (e.g. 2) variables : $z = f(x,y)$

(*) **Extrema** : (x_0, y_0, z_0) so that $\left.dz\right|_{\substack{x=x_0\\y=y_0}} = 0$

But, since $\left.dz\right|_{\substack{x=x_0\\y=y_0}} = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = 0$ $\left.\frac{\partial z}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} = 0$ and $\left.\frac{\partial z}{\partial y}\right|_{\substack{x=x_0\\y=y_0}} = 0$

$\Rightarrow Df = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ The gradient is a null vector

* **Maxima** :
  - Sufficient condition : $\left.d^2z\right|_{\substack{x=x_0\\y=y_0}} < 0$
  - Necessary condition : $\left.d^2z\right|_{\substack{x=x_0\\y=y_0}} \leq 0$

* **Minima** :
  - Sufficient condition : $\left.d^2z\right|_{\substack{x=x_0\\y=y_0}} > 0$
  - Necessary condition : $\left.d^2z\right|_{\substack{x=x_0\\y=y_0}} \geq 0$

Explicitly,

- The second differential d^2z is given by:

$$d^2z = d(dz) = \frac{\partial}{\partial x}(dz) + \frac{\partial}{\partial y}(dz)\,dy$$

But $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dx$

$$\Rightarrow d^2z = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) =$$

$$= \frac{\partial^2 z}{\partial x^2}dx^2 + \frac{\partial^2 z}{\partial x \partial y}dy\,dx + \frac{\partial^2 z}{\partial y \partial x}dy\,dx + \frac{\partial^2 z}{\partial y^2}dy^2$$

* Maxima:

- Sufficient condition: $d^2z\Big|_{\substack{x=x_0\\ y=y_0}} < 0 \Rightarrow \frac{\partial^2 z}{\partial x^2}dx^2 + \frac{\partial^2 z}{\partial x \partial y}dy\,dx + \frac{\partial^2 z}{\partial y \partial x}dy\,dx + \frac{\partial^2 z}{\partial y^2}dy^2 > 0$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} < 0,\quad \frac{\partial^2 z}{\partial y^2} < 0 \quad \text{and} \quad \underbrace{\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 < \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2}}_{\text{this guarantees that the surface will have the same type of configuration in all directions}}$$

at (x_0, y_0, z_0).

$\Rightarrow$ Hessian Matrix is NEGATIVE DEFINITE

(*) Reminder: A matrix $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ is negative definite if

$$M_{11} < 0,\ M_{22} < 0 \ \&\ \mathrm{Det}\,M > 0$$

$$(M_{11}M_{22} > M_{21}M_{12})$$

- Hessian Matrix. $D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$ (by comparison one obtains the above-stated conditions)

- Necessary condition: $d^2z \leq 0 \Rightarrow$ Hessian matrix is ~~positive~~ negative semidefinite ($M_{11} \leq 0$, $M_{22} \leq 0$; $M_{11}M_{22} \geq M_{21}M_{12}$)

* Minima

- Sufficient condition: $d^2z\Big|_{x=x_0,\, y=y_0} > 0$
Hessian matrix is POSITIVE DEFINITE
- Necessary condition: $d^2z \geq 0$: Hessian matrix is positive semidefin

Example: Find and classify the critical points of the function $f(x,y) = x^3 + y^2 - 4xy - 3x$

Critical points: $\dfrac{\partial f(x,y)}{\partial x} = 0$

$$\frac{\partial f}{\partial y}(x,y) = 0$$

$$\frac{\partial f}{\partial x} = 3x^2 - 4y - 3 = 0 \ (*) \ ; \ \frac{\partial f}{\partial y} = 2y - 4x = 0 \quad (**)$$

$$\therefore y = 2x$$

Inserting into $(*)$:

$$3x^2 - 8x - 3 = 0 \Rightarrow x^2 - \frac{8}{3}x - 1 = 0$$

$$x = \frac{4}{3} \pm \frac{1}{2}\sqrt{\left(\frac{8}{3}\right)^2 + 4}$$

$$x = \frac{4}{3} \pm \frac{1}{6}\sqrt{\underbrace{8^2 + 4 \times 9}_{100}}$$

$$x = \frac{4}{3} \pm \frac{\cancel{10}\,5}{\cancel{6}\,3} \quad x_1 = 3, \quad x_2 = -1/3$$

Finding y, $f(x,y)$

(a) $y_1 = 2x_1 = 6$, $f(x,y) = -18$

(b) $y_2 = -\frac{2}{3}$, $f(x,y) = \frac{14}{27}$

: Hessian matrix $D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -4$$

$$D^2 f = \begin{pmatrix} 6x & -4 \\ -4 & 2 \end{pmatrix}$$

(a) $D^2f = \begin{pmatrix} 18 & -4 \\ -4 & 2 \end{pmatrix}$ $\Rightarrow$ Positive definite: $x_0 = 3$, $y_0 = 6$ determine a minimum

(b) $D^2f \begin{pmatrix} -2 & -4 \\ -4 & 2 \end{pmatrix}$ $\Rightarrow$ none of the cases specified (diagonal entries have different signs)

$x_0 = -\frac{1}{3}$, $y_0 = -\frac{2}{3}$ determine a SADDLE POINT

(note that $\left.\frac{\partial^2 f}{\partial x^2}\right|_{\substack{x=x_0\\y=y_0}} < 0$ and $\left.\frac{\partial^2 f}{\partial y^2}\right|_{\substack{x=x_0\\y=y_0}} > 0$)

(*) Please note: • If $Det[D^2f] < 0$ the critical point is a saddle point

• If $Det[D^2f] = 0$ the situation is ambiguous and has to be checked

Example: Classify the critical point (0,0,0) of

$f(x,y) = 2x^2 + 2y^2 - 4xy - x^4 - y^4$

Gradient: $Df = \begin{pmatrix} 4x - 4y - 4x^3 \\ 4y - 4x - 4y^3 \end{pmatrix}$

Hessian matrix $D^2f = \begin{pmatrix} 4-12x^2 & -4 \\ -4 & 4-12y^2 \end{pmatrix}$

$(0,0,0) \Rightarrow Df = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (critical point)

$D^2f = \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}$ Positive semidefinite:

$Det[D^2f] = 0$

Diagonal entries > 0

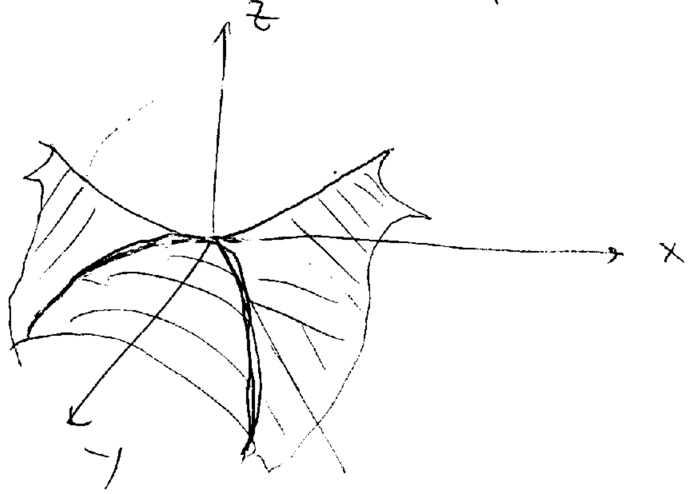
$\Rightarrow$ can be anything! (12)

Ad hoc analysis: if $y=0 \Rightarrow f(x,0) = 2x^2 - x^4$ $\Rightarrow$ has a min at $x=0$

$+ \left.\frac{d^2 f}{dx^2}\right|_{y=0} = 4 - 4 \cdot 3x^2$

if $y = x \Rightarrow f(x,x) = -2x^4 \Rightarrow$ has a max. at $x=0$

$\Rightarrow (0,0,0)$ is a saddle point!



4.3 - Global Optima

- $f(x,y)$ attains a global maximum at (x_0, y_0) if $f(x_0,y_0) \geq f(x,y)$ $\forall (x,y) \in \mathbb{R}^2$
- $f(x,y)$ attains a global minimum at $f(x_0,y_0) \leq f(x,y)$ $\forall (x,y) \in \mathbb{R}^2$

4.3.2 - Concave functions:

- The function $f(x,y)$ is concave if & only if $D^2 f(x,y)$ is negative semidefinite for all (x,y)
- The concave function attains a global maximum at (x_0,y_0) if & only if $Df(x_0,y_0) = 0$

Example: Show that $f(x,y) = 1-(y-1)^2-(x+1)^2$ is a concave function

- 1st Step: Find the Hessian Matrix

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-2(x+1)\right) = -2$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(-2(y-1)\right) = -2$$

$$\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x = 0$$

$$D^2 f = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow$$ NEGATIVE DEFINITE : the function is convex

Maxima at $-2(x+1) = 0 \Rightarrow x = -1$

$-2(y-1) = 0 \Rightarrow y = 1$ $\Rightarrow f(x,y) = 1$