SECOND-ORDER DIFFERENTIAL EQUATIONS

I. Definition: a second-order differential equation is a differential equation involving second derivatives; i.e.,

~~Exa~~ $$\frac{d^2y}{dt^2} = F\left(t, y, \frac{dy}{dt}\right) \quad (1)$$

(we wish to find all possible functions $y = f(t)$ which satisfy (1))

Examples: $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 1 = 0$; $\frac{d^2y}{dt^2} + \sin y \frac{dy}{dt} + 2y^3 = t^2$

(*) We will restrict ourselves to linear equations with constant coefficients

1. Homogeneous equations

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad (*)$$

- Guess: we will look for solutions of the form $y = e^{\alpha t}$ (α = const)
(this guess is based on the fact that a linear 1^{st}-order differential equation with constant coefficients has a solution of this form)

Hence; $\frac{d^2y}{dt^2} = \alpha^2 e^{\alpha t}$; $\frac{dy}{dt} = \alpha e^{\alpha t}$

- Inserting into (*) $\Rightarrow$ $\alpha^2 e^{\alpha t} + b\alpha e^{\alpha t} + c e^{\alpha t} = 0 \Rightarrow \underbrace{(\alpha^2 + b\alpha + c)}_{(**)} e^{\alpha t} = 0$

The above-stated condition is only satisfied if $(**) = 0$
$\Rightarrow$ to solve (*) we need to look for the roots of (**)
~~Pleas~~

(*) Please note: the quadratic equation (**) is called the characteristic equation of the differential equation (*) ($y = e^{\alpha t}$ is a solution of (*) only if ~~α~~ is a solution of (**))

General case: $\alpha = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$ (Baskara's formula; from A-levels)

1.1 - Distinct roots : $\sqrt{b^2 - 4c} \neq 0$

$$\alpha_1 = \frac{-b+\sqrt{b^2-4c}}{2} \quad ; \quad \alpha_2 = \frac{-b-\sqrt{b^2-4c}}{2}$$

$$y(t) = A e^{\alpha_1 t} + B e^{\alpha_2 t}$$

(*) Please note: in order to find the general solution of a second-order differential equation one needs two arbitrary constants: this comes from the fact that one is integrating twice in this case.

1.1.1 - Real roots: $b^2-4c>0$

$y(t) = A e^{\alpha_1 t} + B e^{\alpha_2 t}$ ~~is~~ are exponentially increasing / decaying : if $\alpha_1<0$ AND $\alpha_2<0$: y decays exponentially; in all other cases y increases exponentially

Example: Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 6y = 0$$

- 1st step: $y = e^{\alpha t} \Rightarrow (\alpha^2+\alpha-6)e^{\alpha t} = 0 \Rightarrow$ the characteristic equation is $\alpha^2+\alpha-6=0$ (**)
- 2nd step: find the roots of (**) $\Rightarrow$ $\alpha_1 = 2$, $\alpha_2 = -3$

Hence $y(t) = A e^{2t} + B e^{-3t}$

1.1.2 - Complex roots $b^2-4c<0$. We will write

$$\alpha_1 = -\frac{b}{2} + \frac{1}{2}\sqrt{-1\,\underbrace{(4c-b^2)}_{\beta>0}} \quad ; \quad \alpha_2 = -\frac{b}{2} - \frac{1}{2}\sqrt{-1\,\underbrace{(4c-b^2)}_{\beta>0}}$$

But $\sqrt{-1} = i$ so that $\alpha_1 = -\frac{b}{2} + \frac{i}{2}\beta$; $\alpha_2 = -\frac{b}{2} - \frac{i}{2}\beta$

Hence $y(t) = A e^{-\frac{b}{2}t} e^{i\beta/2} + B e^{-b/2\,t} e^{-i\beta/2} = e^{-b/2\,t}\left(A e^{i\beta t/2} + B e^{-i\beta/2\,t}\right)$

(*) Please note: $e^{\pm i\beta t}$ can be written in terms of $\cos\beta t$, $\sin\beta t$

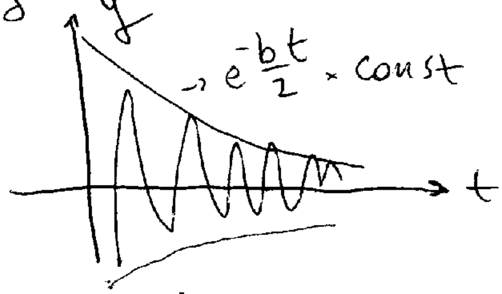
Proof: $e^{\pm i\beta t} = \cos\beta t \pm i\sin\beta t$

$$y(t) = e^{-\frac{b}{2}t}\left[A(\cos\beta t + i\sin\beta t) + B(\cos\beta t - i\sin\beta t)\right]$$

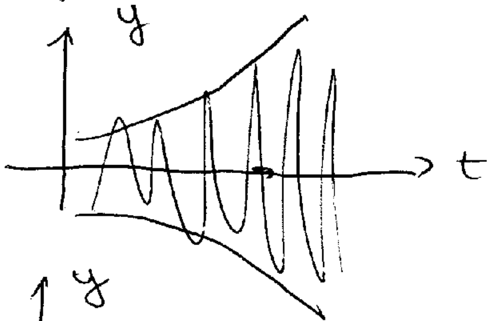
$$y(t) = e^{-\frac{b}{2}t}\left[\underbrace{(A+B)}_{C}\cos\beta t + \underbrace{i(A-B)}_{D}\sin\beta t\right]$$

These solutions are oscillatory

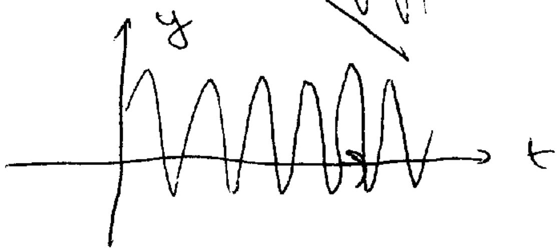
- $b>0$: damped oscillations



- $b<0$: explosive oscillations:



- $b=0$: oscillations



Example: (a) Find the general solution of $\frac{d^2y}{dt^2} + 4y = 0$

Ansatz: $y = e^{\alpha t}$ so that $\frac{d^2y}{dt^2} = \alpha^2$; $(\alpha^2+4)e^{\alpha t} = 0$

Characteristic equation $\alpha^2 + 4 = 0$

$(\alpha - 2i)(\alpha + 2i) = 0$ Roots: $\alpha_1 = 2i$, $\alpha_2 = -2i$

$$\Rightarrow \boxed{y = Ae^{2it} + Be^{-2it}}$$

(b) find the specific solution of this equation so that $y(0)=0$ and $\frac{dy}{dt} = 1$ when $t=0$.

(*) Please note: in order to find a specific solution of a 2^{nd}-order diff. eq, one needs two boundary/initial conditions

$y(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A$

$$\left.\frac{dy}{dt}\right|_{t=0} = 1 \Rightarrow \frac{dy}{dt} = \left.\left(2iAe^{2it} - 2iBe^{-2it}\right)\right|_{t=0} = 1 \Rightarrow \underbrace{2iA - 2iB}_{4iA} = 1$$

$A = \frac{1}{4i}$, $B = \frac{-1}{4i}$

$$y(t) = \frac{1}{4i} e^{2it} - \frac{1}{4i} e^{-2it} = \frac{1}{2}\left[\underbrace{\frac{e^{2it} - e^{-2it}}{2i}}_{\sin(2t)}\right]$$

1.2 - Coincident roots

If, in the characteristic equation, $b^2 = 4c$ then $\alpha = \alpha_1 = \alpha_2 = -\frac{b}{2}$. Hence $y(t) = A e^{-\frac{b}{2}t} + f(t)$.

$f(t) = Bte^{\alpha t}$ is also a solution of the differential equation.

Proof: Let us consider $\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$ with the root $\alpha = -\frac{b}{2}$ and $b^2 = 4c$.

If $f(t)$ is a solution of this equation, then

$$\frac{d^2 f(t)}{dt^2} + b\frac{df(t)}{dt} + c f(t) = 0 \qquad (*)$$

$$\frac{df(t)}{dt} = B(e^{\alpha t} + \alpha t e^{\alpha t})$$

$$\frac{d^2 f(t)}{dt} = B(\alpha e^{\alpha t} + \alpha^2 t e^{\alpha t} + \alpha e^{\alpha t})$$

Inserting in $(*)$ $\Rightarrow B\left[(2\alpha + \alpha^2 t)e^{\alpha t} + b(e^{\alpha t} + \alpha t e^{\alpha t}) + cte^{\alpha t}\right] = 0$

Since $Be^{\alpha t} \neq 0$, $(2\alpha + \alpha^2 t + b + b\alpha t + ct) = 0$

$$\Rightarrow (\underbrace{2\alpha + b}_{} + \underbrace{(\alpha^2 + b\alpha + c)}_{(**)}t) = 0$$

$$2\alpha + b = 2 \cdot \frac{-b}{2} + b = 0$$

$$\alpha^2 + b\alpha = \frac{b^2}{4} + b\cdot\left(\frac{-b}{2}\right) = \frac{-b^2}{4} = -c \text{ so that } (**) = 0.$$

Example : Find the general solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Characteristic equation: $\alpha^2 - 2\alpha + 1 = 0$

$$(\alpha - 1)^2 = 0$$

$$\alpha = 1$$

Hence $y = e^t [B + At]$

2. Non-homogeneous equations

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = G(t)$$

How to solve this equation?

Method : • Find a particular solution $y_p(t)$ so that

$$\frac{d^2 y_p}{dt} + b\frac{dy_p}{dt} + cy_p = G(t)$$

(Guess: this particular solution is assumed to be of the same form as $G(t)$)

• Set $z = y - y_p$ and find the general solution of the associated differential equation

$$\frac{d^2z}{dt^2} + b\frac{dz}{dt} + cz = 0$$

Example : Find the general solution of the differential equation

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 9y = 1 + t$$

$y = y_p + z$ (particular solution + solution of associated h. eq.)

• Particular solution : $y_p = a + bt$

$$\frac{dy_p}{dt} = b \quad \Rightarrow \quad 6b + 9(a + bt) = 1 + t$$

$$\frac{d^2y_p}{dt^2} = 0 \qquad 6b + 9a = 1$$

$$9b = 1$$

$$b = \frac{1}{9}; \quad \frac{\cancel{6}^2}{\cancel{9}_3} + 9a = 1$$

$$9a = \frac{1}{3}; a = \frac{1}{27}$$

$$y_p = \frac{1}{27} + \frac{1}{9}t$$

Complementary solution : z so that

$$\frac{d^2z}{dt^2} + 6\frac{dz}{dt} + 9z = 0$$

$$z = e^{\alpha t} \quad \text{so that} \quad (\alpha^2 + 6\alpha + 9)e^{\alpha t} = 0$$

$$\alpha = \frac{-3 \pm \sqrt{36 - \underbrace{4.9}_{36}}}{2} = \alpha = -3$$

This equation has coincident roots so its general solution is $z = (A + Bt)e^{-3t}$

Finaly $\boxed{y = (A + Bt)e^{-3t} + \frac{1}{27} + \frac{1}{9}t}$