6. Principles of constrained optimization

6.1 Introduction

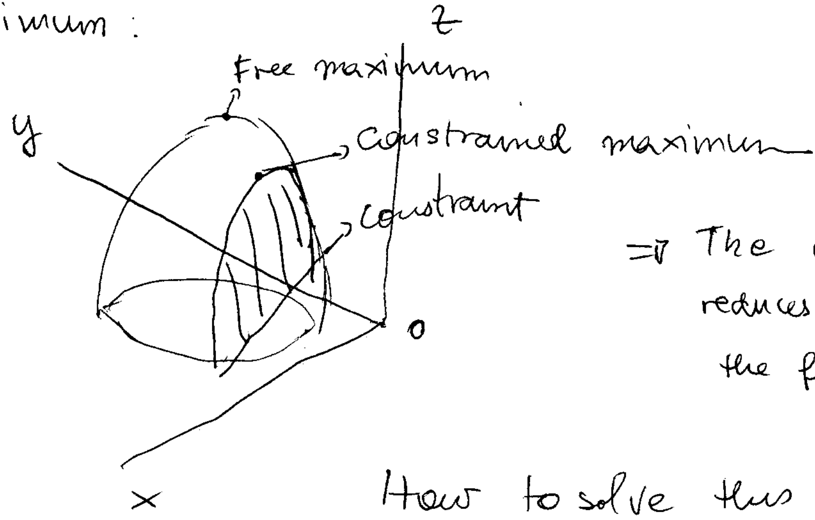
⊛ Problem: Let $f(x,y)$ and $g(x,y)$ be functions of two variables. We wish to maximize $f(x,y)$ subject to the constraint $g(x,y)=0$

Examples:

- Maximize $4xy-2x^2+y^2$ subject to $3x+y=5$
  ($f(x,y)=4xy-2x^2+y^2$; $g(x,y)=3x+y-5$

- Maximize $U(x_1,x_2)=x_1x_2+2x_1$ subject to the constraint
  $4x_1+2x_2=60$

⊛ Difference between a constrained maximum and a free maximum:



$\Rightarrow$ The constraint reduces the domain of the function

How to solve this problem?

~~6.2 - Lagrange multipliers~~

[illegible] $f(x,y)$ and the constraint $g(x,y)$

- Possibility 1:

- Solve $g(x,y)=0$ for x $\Rightarrow$ $y=h(x)$
- Substitute into $f(x,y)$:
  $F(x)=f(x,h(x))$
- Find the maximum of $F(x)$

(Not always possible: $g(x,y)=0$ may be messy)

$\Rightarrow$ We need the techniques seen for implicit relations

<u>Possibility 2</u> : Lagrange multipliers

Applying the chain rule :

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \qquad (*)$$

How to find $\frac{dy}{dx}$?

$y = h(x)$ satisfies the implicit relations $g(x,y) = 0$

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-\partial g/\partial x}{\partial g/\partial y}$$

Substituting in $(*)$:

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial f}{\partial y} \times \frac{(\partial g/\partial x)}{\partial g/\partial y}$$

x : $\frac{dF}{dx} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\left(\frac{\partial g}{\partial x}\right)\Big/(\partial g/\partial y) = 0$

calling $\lambda = \frac{\partial f/\partial y}{\partial g/\partial y}$ $(**)$ we have $\boxed{\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0}$

This gives

$$\frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0$$

$\Rightarrow$ First-order conditions for a constrained maximum

$$\frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 \quad (\text{from } (**))$$

6.2 - <u>Lagrange multipliers</u>

$(*)$ We will provide a systematic way of obtaining the 1^{st}-order conditions

- Step 1 : Define a function L of 3 variables

$$L(x,y,\lambda) = f(x,y) - \lambda\, g(x,y)$$

$L \equiv$ "Lagrangian function"

$\lambda \equiv$ "Lagrange multiplier".

$\Rightarrow$ The conditions for a critical point of L are

$$\left.\begin{aligned} \frac{\partial L}{\partial x} = 0 &\Rightarrow \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = 0 &\Rightarrow \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 \end{aligned}\right\} \text{first-order conditions}$$

$$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow g(x,y) = 0 \longrightarrow \text{constraint}$$

- Step 2 : Solve the 1st-order conditions for x, y in terms of λ
- Step 3 : Use $g(x,y) = 0$ to solve for λ^2.

Example: Find the extremum of

$$\underbrace{f(x,y) = xy}_{\text{function}} \quad \text{subject to} \quad \underbrace{x+y=6}_{\text{constraint}}$$

- Lagrangian function :

$$L(x,y,\lambda) = xy + \lambda(6 - x - y)$$

Stationary values of L :

$$\frac{\partial L}{\partial x} = y - \lambda = 0 \Rightarrow y = \lambda$$

$$\frac{\partial L}{\partial y} = x - \lambda = 0 \Rightarrow x = \lambda$$

$$\frac{\partial L}{\partial \lambda} = 6 - x - y = 0$$

$\Rightarrow 6 - 2\lambda = 0$

$\lambda = 3$

$x = 3$

$y = 3$

nstrained Max. : $(3, 3, 9)$

(*) Please note :

- Lagrange's method can be extended to functions of more than 2 variables:

$$\underbrace{L(x_1, x_2, \dots x_n)}_{\text{Lagrangian function}} = \underbrace{f(x_1, x_2, \dots, x_n)}_{\text{function}} - \lambda \underbrace{g(x_1, x_2, \dots, x_n)}_{\text{Constraint}}$$

1st-Order Conditions $\frac{\partial L}{\partial x_1} = 0$, $\frac{\partial L}{\partial x_2} = 0 \dots \frac{\partial L}{\partial x_n} = 0$

constraint : $\frac{\partial L}{\partial \lambda} = 0$

- One may have more than one constraint :

Example : to maximize $f(x, y, z, w)$ subject to

$g(x, y, z, w) = 0$ and $h(x, y, z, w) = 0$

we write

$$L(x, y, z, w, \lambda, \mu) = \underbrace{f(x, y, z, w)}_{\text{function}} - \lambda \overbrace{g(x, y, z, w)}^{\text{constraints}} - \mu \overbrace{h(x, y, z, w)}^{\text{constraints}}$$

Lagrange multipliers

- This method only works if the implicit differentiation methods can be applied.