Integration

I- <u>Definition</u> Integration is the reverse of differentiation.

- Let us consider a function $f(x) = \frac{dF(x)}{dx}$. Then $F(x) = \int f(x)\,dx$ (the integral of $f(x)$ with respect to x).

- <u>Example</u>: $F(x) = x^2 + \text{Const}$

Differentiation: $f(x) = \frac{dF(x)}{dx} = \underbrace{\frac{d}{dx}(x^2)}_{2x} + \underbrace{\frac{d}{dx}(\text{Const})}_{0} \Rightarrow f(x) = 2x$

Integration: $\int 2x\,dx = x^2 + \text{Const}$.

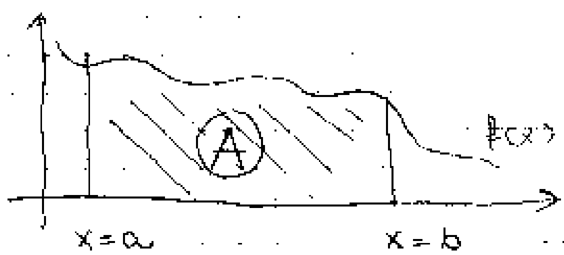
* Please note: here we will consider CONTINUOUS FUNCTIONS, unless otherwise stated.

II- Areas and integrals

1. Geometrical (intuitive) interpretation: integration is a method of computing areas in a plane.

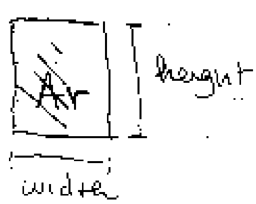
- <u>Example</u>: How to compute the area under a function $f(x)$, in the interval $x \in [a, b]$?

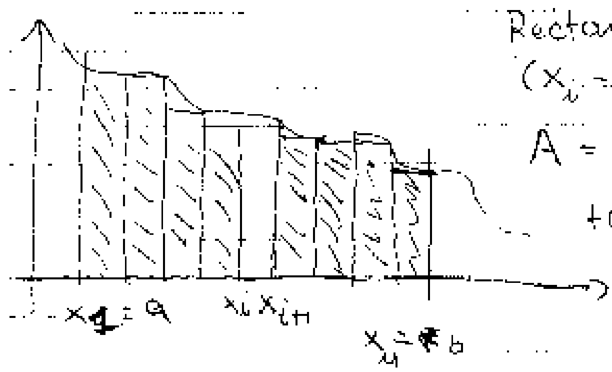
<u>Problem</u>:



<u>Approximation</u>: We will divide (A) into areas we know how to compute (for instance rectangles).

Area of a rectangle
A_r = height × width





Rectangles have equal widths. ($x_i - x_{i-1} = \Delta x$)

$A = (x_2 - x_1)\,f(x_2) + (x_3 - x_2)\,f(x_3) + \dots + (x_n - x_{n-1})\,f(x_n) = \sum_{i=1}^{n} f(x_i)\,\Delta x$

②

- The smaller Δx is, the better the approximation
- $\lim_{\Delta x \to 0} \sum_{i=1}^{n} \Delta x\, f(x_i) = \int_a^b f(x)\,dx$ [integral of f from a to b)

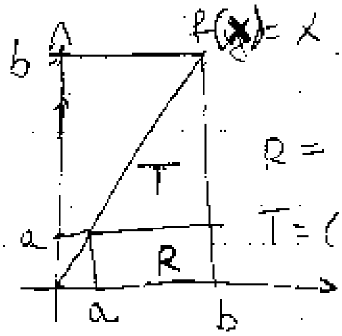
What happens if f(x) lies below the x axis?

Convention: "areas" below the x axis are negative

$$\int_a^b f(x)\,dx < 0$$

Examples

$$f(x) = x \Rightarrow \int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$$



$R = (b-a) \times a$

$T = \frac{(b-a)^2}{2}$

$$A = (b-a) \times a + \frac{(b-a)^2}{2} = \frac{1}{2}(b^2 - a^2)$$

$$\underbrace{(b-a)\left(a + \frac{b-a}{2}\right)}_{} = \frac{1}{2}(a+b)(b-a)$$

What does this have to do with differentiation?

III - Integration and differentiation

1. Fundamental theorem of calculus

Given a real number a and a continuous function $f(t)$, we can define another function $F(t)$ as

$$F(t) = \int_a^t f(x)\,dx \quad \text{for all } t$$

Then, $F(t)$ is differentiable and

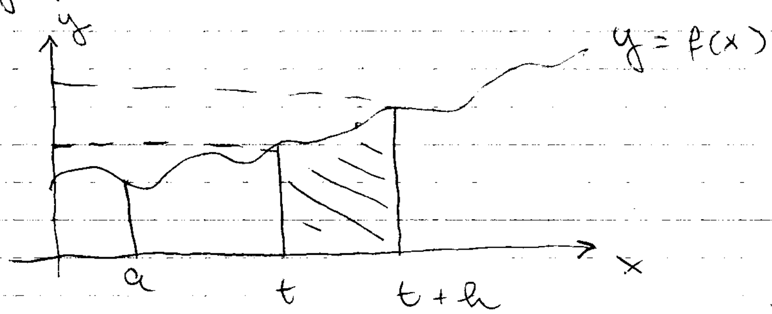
$$F'(t) = f(t) \quad \forall\ t$$

(*) More rigorous statement about integration being the reverse of differentiation

(see beginning of lecture)

Proof:

- Let us consider the area under a continuous function $y = f(x)$ between $x = t$ and $x = t + h$

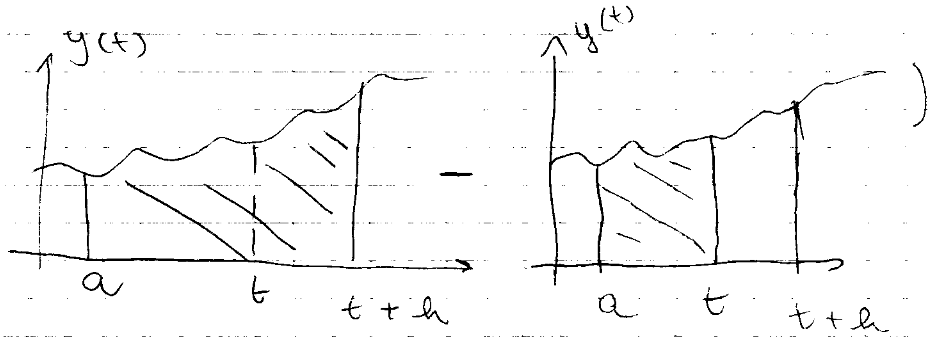


- This area is given by $A = \int_t^{t+h} f(x)\,dx$ (see previous section)

- This area can also be written as

$$A = \int_a^{t+h} f(x)\,dx - \int_a^t f(x)\,dx$$

(i.e., as the difference



)

From the above-stated definition, $A = F(t+h) - F(t)$

- Let M be the greatest value of $f(x)$ in $[t, t+h]$ and m the least value of $f(x)$ in $[t, t+h]$

- Then $mh \leq F(t+h) - F(t) \leq Mh \Rightarrow m \leq \frac{F(t+h) - F(t)}{h} \leq M$

- small h. $f(x)$, $x \in [t, t+h]$ are close to $f(t)$

$f(t)$ is close to m, M (expression above gets "squeezed")

$$\boxed{\Rightarrow f(t) = \lim_{h \to 0} \frac{F(t+h) - F(t)}{h} =}$$

(definition of derivative)

2. Integrating derivatives

If the function $g(x)$ has a continuous derivative,

$$\int_a^b g'(x)\,dx = g(b) - g(a)$$

Proof:

$$I = \int_a^b g'(x)\,dx = g(b) - g(a)$$

$$F(t) = \int_a^t g'(x)\,dx \quad \text{then} \quad F(a) = 0$$
$$F(b) = I$$

From the fundamental theorem of calculus $g'(t) = F'(t)\ \forall t$

Then $F'(t) - g'(t) = 0 \Rightarrow \frac{d}{dt}\left(F(t) - g(t)\right) = 0$

The derivative of a constant always vanishes, so that

$F(t) - g(t) = \text{const.}$

Hence $F(b) - g(b) = F(a) - g(a) \Rightarrow F(b) - F(a) = g(b) - g(a)$

$$\Rightarrow \boxed{g(b) - g(a) = I - 0 = I}$$

3. Primitives

• Definition: Let $f(x)$ be a continuous function. A primitive or anti-derivative of $f(x)$ is a function $g(x)$ such that

$g'(x) = f(x)$ for all x

Please note: to compute $\int_a^b f(x)\,dx$ we only need to find a primitive of f

Example 1

(a) Find a primitive of $f(x) = x^2$

(b) Compute $\int_1^2 x^2\,dx$

(a) $\frac{d}{dx}\left(\frac{x^3}{3}\right) = x^2$, so the primitive is $g(x) = \frac{x^3}{3}$

(b) $\int_a^b g'(x)\,dx = g(b) - g(a)$

In our case, $g'(x) = f(x) = x^2$ $\quad a = 1$

$\quad b = 2$

$g(x) = \frac{x^3}{3}$

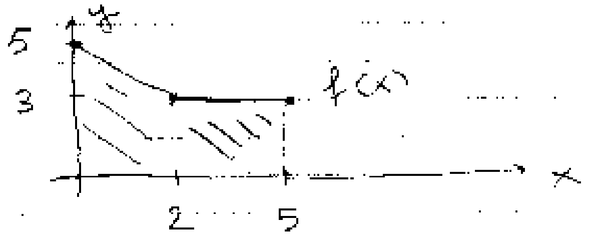
Hence $\displaystyle\int_1^2 x^2 dx = \frac{x^3}{3}\bigg|_{x=2} - \frac{x^3}{3}\bigg|_{x=1} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3} = 2{,}333$

Example 2 : Compute $\displaystyle\int_0^5 f(x)dx$ with

$$f(x) = \begin{cases} 5-x & x \leq 2 \\ 3 & x > 2 \end{cases}$$

(Please note: this function is discontinuous, but in some specific cases they can be integrated)

• How does f look like?



• We are interested in the area below $f(x)$.

Possibility 1 compute the area of 2 rectangles + a triangle and sum the areas

Possibility 2 use what we have learned about primitive function

Method Divide the integral into 2 regions:

$$I = \int_0^5 f(x)dx = \underbrace{\int_0^2 (5-x)dx}_{I_1} + \underbrace{\int_2^5 3\,dx}_{I_2}$$

$$I_1 = \int_0^2 (5-x)\,dx = g_1(2) - g_1(0)$$ ($g_1(x)$ is the primitive function)

• Find the primitive : $\displaystyle\frac{d}{dx}\left(5x - \frac{x^2}{2}\right) = 5 - x \Rightarrow$ Primitive is $g_1(x) = 5x - \dfrac{x^2}{2}$

$g_1(0) = 0$

$g_1(2) = 5 \times 2 - \dfrac{(2)^2}{2} = 10 - 2 = 8$

$$\Rightarrow \boxed{I_1 = g_1(2) - g_1(0) = 8 - 0 = 8}$$

$$I_2 = \int_2^5 3dx = g_2(5) - g_2(2) \quad (g_2 \text{ is the primitive})$$

$$\frac{d}{dx}(3x) = 3 \Rightarrow g_2(x) = 3x$$

$$I_2 = g_2(5) - g_2(2) = 15 - 6 = 9$$

$$I = I_1 + I_2 = 8 + 9 = 17$$

⊛ Please note:

- A function does not have a unique primitive (in fact, there is a whole set of them!)

  Example $\quad g_1(x) = x^2 + 2$ and $g_2(x) = x^2 - 5$ are primitives of $f(x) = 2x$

  This comes from the fact that the derivative of a constant vanishes. Hence, all the primitives of $f(x)$ are of the form $g(x) = x^2 + \underbrace{\text{Const}}_{\text{arbitrary constant}}$,

- Depending on the function we wish to integrate, finding a primitive may not be straightforward $\Rightarrow$ one needs systematic way of finding primitives.

IV-Rules of integration

1. Indefinite Integral:

1.1 Definition: The indefinite integral of a function $f(x)$ is the set of all its primitives

1.2 Notation: $\int f(x)\,dx = \underbrace{g(x)}_{\text{particular primitive}} + \underbrace{C}_{\text{arbitrary constant "constant of integration"}}$

the function which is being integrated is called "the integrand".

1.3 Rules for computing indefinite integrals

(a) $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C \quad , (\alpha \neq -1)$ {The power rule}

comes from the fact that $\frac{d}{dx}\left(\frac{x^{\alpha+1}}{\alpha+1}\right) = \frac{\cancel{(\alpha+1)}\, x^{\cancel{\alpha+1-1}}}{\cancel{(\alpha+1)}}$

Examples

$$\int x^3 dx = \frac{x^4}{4} + C \quad (n=3)$$

$$\int 1\, dx = x + C$$ (note that $x^0 = 1$ so that $n=0$)

$$\int \sqrt{x^3}\, dx = \frac{x^{5/2}}{5/2} + C = \frac{2}{5} x^{5/2} + C$$ (note that $\sqrt{x^3} = x^{3/2}$ so that $n = 3/2$)

(b) $\int e^x dx = e^x + C$ (the exponential rule)

(c) $\int \frac{1}{x} dx = \ln x + C$ $(x > 0)$ (the logarithmic rule)

(one should note that logarithms do not exist for $x \leq 0$)

1.4 - Rules of operation

(a) The integral of a sum of a finite number of functions is the sum of the integrals of those functions

PARTICULAR (SIMPLEST) CASE: 2 functions

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

Proof: Let us consider $f(x) = \frac{d}{dx} F(x)$

$$g(x) = \frac{d}{dx} G(x)$$

From differential calculus we know that

$$\frac{d}{dx}[F(x) + G(x)] = \frac{d}{dx} F(x) + \frac{d}{dx} G(x) = f(x) + g(x)$$

$$\int [f(x) + g(x)]\,dx = F(x) + G(x) + C$$

$$\int f(x)\,dx = F(x) + C_1 \quad ; \quad \int g(x)\,dx = G(x) + C_2$$

$$\int f(x)\,dx + \int g(x)\,dx = F(x) + G(x) + \underbrace{C_1 + C_2}$$

can be set as C, since both constants are arbitrary

$$\Rightarrow \boxed{\int f(x)\,dx + \int g(x)\,dx = \int [f(x) + g(x)]\,dx}$$

<u>Example</u> : Find $I = \int [x + e^x + 1]\,dx$

- According to the above-stated rule,

$$I = \underbrace{\int x\,dx}_{I_1} + \underbrace{\int e^x\,dx}_{I_2} + \underbrace{\int 1\,dx}_{I_3}$$

$$I_1 = \int x\,dx = \frac{x^2}{2} + C_1 \quad (\text{"power rule"})$$

$$I_2 = \int e^x\,dx = e^x + C_2 \quad (\text{exponential rule})$$

$$I_3 = \int dx = \int x^0\,dx = x + C_3 \quad (\text{power rule})$$

$$\Rightarrow I = \frac{x^2}{2} + C_1 + e^x + C_2 + x + C_3 = \frac{x^2}{2} + e^x + x + C$$

(b) The integral of k times a function ~~an integrand~~ (k being a constant) is k times the integral of that function

$$\int k f(x)\,dx = k \int f(x)\,dx$$

Proof: $\int k f(x)\,dx = \int [\underbrace{f(x) + f(x) + \ldots + f(x)}_{k \text{ terms}}]\,dx$

Using the sum rule,

$$\int [\underbrace{f(x)+\dots+f(x)}_{k \text{ terms}}]\,dx = \underbrace{\int f(x)dx + \dots + \int f(x)dx}_{k \text{ terms}} = k\int f(x)\,dx$$

SUMMARY : RULES OF INTEGRATION

I - Indefinite integrals of specific functions

1. The power rule $\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C \; , \; \alpha \neq -1$

2. The exponential rule : $\int e^{x} dx = e^{x} + C$

3. The logarithmic rule : $\int \frac{dx}{x} = \ln x + C \; (x>0)$

4. Trigonometric integrals :

   4.a $\int \cos x\, dx = \sin x + C$

   4.b $\int \sin x\, dx = -\cos x + C$

II Rules of operation

1. Integral of the sum : $\int [f(x)+g(x)]\,dx = \int f(x)dx + \int g(x)\,dx$

2. Integral of a multiple : $\int k f(x) dx = k \int f(x)\,dx$

2. Definite Integral

An integral of the form $\int_a^b f(x)\,dx$ is a definite integral

- The numbers a, b are the limits of integration
- The interval $[a, b]$ is the range of integration

(*) Such integrals can be interpreted as the area under a curve

Method for computing definite integrals

(a) Compute the indefinite integral $\int f(x)\,dx = F(x) + C$

(b) Choose a particular primitive (usually with $C \equiv 0$)

(c) Compute the definite integral as $F(b) - F(a)$

Please note: $F(b) + C - [F(a) + C] = F(b) - F(a)$

(the arbitrary constant cancels out)

Consequence: one may choose ANY primitive for computing (c), but it is convenient to choose the easiest F

IV. Further methods of integration

1. The substitution method (integration by change of variable)

(*) this is the integral-calculus counterpart of the composite rule

Let us consider $u = u(x)$ and a function $f = f(u)$.

The integral of $f(u)\frac{du}{dx}$ with respect to the variable x is the integral of $f(u)$ with respect to the variable u.

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du = F(u) + C$$

(the operation $\int dx$ has been substituted by $\int du$)

$$\frac{d}{dx}F(u) = \frac{d}{du}F(u)\frac{du}{dx} = F'(u)\frac{du}{dx} = f(u)\frac{du}{dx}$$

Since $f(u)\frac{du}{dx}$ is the derivative of $F(u)$, its integral must be

$$\int f(u)\frac{du}{dx}\,dx = F(u) + C$$

If one "cancels" the dx on the left, one obtains the same expression, i.e.

$$\int f(u)\,du = F(u) + C$$

Hence $\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du = F(u) + C$

Examples:

1. Compute $I = \int \frac{x\,dx}{\sqrt{1-x^2}}$

Let us take $u = 1-x^2$, then $\frac{du}{dx} = -2x \Rightarrow \frac{du}{dx}dx = du = -2x\,dx$

$$I = -\frac{1}{2}\int \frac{(-2x\,dx)}{\sqrt{1-x^2}} = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{2u^{1/2}}{2} + C$$

$$= -\sqrt{1-x^2} + C$$

2. Compute $I = \int \frac{dx}{\sqrt{1-x^2}}$

Calling $x = \sin u$ then $\sqrt{1-x^2} = \sqrt{1-\sin^2 u} = \cos u$

(remember that $\cos^2 u + \sin^2 u = 1$)

and $\frac{du}{dx} = \cos u \Rightarrow dx = \cos u\,du$

$$I = \int \frac{\cos u \, du}{\cos u} = \int du = u + c$$

if $x = \sin u$ then $u = \arcsin x$ $\Rightarrow$ $\boxed{I = \arcsin x + C}$

(*) Please note. in both examples we have written $\frac{du}{dx} = f'(x)$ as $du = f'(x)dx$. This is a notational trick that simplifies the computations (makes the expressions to be integrated easier to visualize)

2. Integration by parts

Let us consider a function $f(x)$. Suppose that this function can be expressed as $f(x) = g(x)h(x)$

Then $f'(x) = g'(x)h(x) + g(x)h'(x)$

and $\int f'(x)dx = f(x) = \int [g'(x)h(x) + g(x)h'(x)]\,dx$

then $g(x)h(x) = \int g'(x)h(x)dx + \int g(x)h'(x)dx$

or $$\boxed{\int g'(x)h(x)dx = g(x)h(x) - \int g(x)h'(x)dx}$$

Example : 1. Compute $\int x e^{-2x} dx$

- First step: choose g, h adequately

we will take $h(x) = x$

$g'(x) = e^{-2x}$

- ~~$g(x)$ we will find~~
- ~~Find $g(x)$~~

• Second step: find $g(x)$: $g(x) = \int e^{-2x}dx$

(performed by substitution method: $u = -2x$

$du = -2dx$

$$\int e^{-2x}dx = -\int e^{u}\frac{du}{2} = \frac{-1}{2}e^{u} = \frac{-1}{2}e^{-2x}$$

• Third step: find $h'(x) = \frac{dx}{dx} = 1$

Then

$$\int \underbrace{g'(x)}\, h(x)dx = g(x)h(x) - \int g(x)h'(x)dx$$

$$I = \int e^{-2x}\cdot x\,dx = -\frac{x}{2}e^{-2x} - \int\left(\frac{-1}{2}\right)e^{-2x}dx = -\frac{x}{2}e^{-2x} + \frac{1}{2}\int e^{-2x}dx$$

$$\Rightarrow \boxed{I = -\frac{x}{2}e^{-2x} + \frac{1}{2}\times\left(\frac{-1}{2}e^{-2x}\right) + C}$$

(*) One always tries to reduce I to an integral we know how to compute

(*) Sometimes one can use integration by parts recursively

2. Compute $\int x^2 e^{-2x}dx$

Let us choose $g'(x) = e^{-2x}$

$h(x) = x^2$

then $g(x) = \frac{-1}{2}e^{-2x}$ (from example 1)

and $h'(x) = 2x$

Hence

$$I = \int x^2 e^{-2x}dx = -\frac{x^2}{2}e^{-2x} - \int 2x\cdot\left(\frac{-1}{2}\right)e^{-2x}dx =$$

$$= -\frac{x^2}{2}e^{-2x} + \underbrace{\int x e^{-2x}dx}_{\text{we know this from Example 1}} = -\frac{x^2}{2}e^{-2x} - \frac{1}{4}e^{-2x} - \frac{x}{2}e^{-2x} + C$$

Example 3: Compute $\int x\cos(ax+b)\,dx$, a, b, constants $\neq 0$

Here we choose $h(x) = x$
$g'(x) = \cos(ax+b)$

$\Rightarrow h'(x) = 1$
$g(x) = \int \cos(ax+b)\,dx$

$\Rightarrow$ substitution of variables: $u = ax+b$
$\frac{du}{dx} = a \Rightarrow du = a\,dx$

$\Rightarrow g(x) = \frac{1}{a}\int \cos u\,du = \frac{1}{a}\sin(ax+b)$

$$\int x\cos(ax+b)\,dx = \frac{x\sin(ax+b)}{a} - \frac{1}{a}\underbrace{\int \sin(ax+b)\,dx}_{= -\frac{1}{a}\cos(ax+b)}$$

$$\Rightarrow \boxed{\int x\cos(ax+b)\,dx = \frac{x\sin(ax+b)}{a} + \frac{1}{a^2}\cos(ax+b)}$$

IV - Improper integrals

1. Infinite limits of integration (also called "infinite integral" sometimes)

- Let us consider the integrals

$$I_1 = \int_a^{\infty} f(x)\,dx \quad \text{and} \quad I_2 = \int_{-\infty}^{b} f(x)\,dx$$

It is not possible to evaluate
$I_1 = F(\infty) - F(a)$ and $I_2 = F(b) - F(-\infty)$ because ∞ is not a number

Procedure:

$$I_1 = \lim_{b\to\infty} \int_a^b f(x)\,dx$$ (I_1 is the limit of a "proper" integral when its upper limit b goes to infinity)

Example : $\int_0^\infty e^{-x}dx = \lim_{b\to\infty} \int_0^b e^{-x} dx =$

$= \lim_{b\to\infty} \left[\left[-e^{-x}\right]_{x=b} - \left[-e^{-x}\right]_{x=0} \right] = \underbrace{1}_{e^{-0}} - \underbrace{\lim_{b\to\infty} e^{-b}}_{=0} = \underline{1}$

The above-stated example is a convergent <u>integral</u>

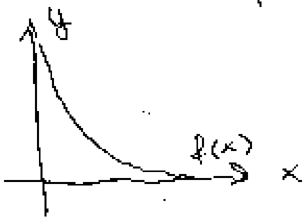
• Definitions (convergent/divergent integrals):

(*) Let us consider a continuous function $f(x)$

- If $\int_a^b f(x)dx$ approaches a finite limit L as $b\to\infty$ the integral converges
- If $\int_a^b f(x)\,dx$ does not approach a finite limit L as $b\to\infty$ the integral diverges

Example of divergent integral $\int_0^\infty \frac{1}{x} dx$

(*) Intuitive interpretation of a convergent integral: even if the integration range is infinite the area under the curve is finite



2 - <u>Infinite integrand</u>: Consider $I = \int_a^b f(x)\,dx$

Even if the limits of integration are finite, an integral can still be improper if ~~the~~ $f(x)$ becomes infinite somewhere within $[a,b]$

<u>Example</u>: 1. $\displaystyle\int_0^1 \frac{dx}{x}$ (this integral is improper because $\displaystyle\lim_{x\to 0^+} \frac{1}{x} = \infty$)

this is indicating that I go towards zero "from the right"

How to compute this integral?

$\ln(1) = 0$

$$\int_0^1 \frac{dx}{x} = \lim_{a\to 0^+} \int_a^1 \frac{dx}{x} = \lim_{a\to 0^+} (-\ln a)$$

This limit does not exist: the integral is DIVERGENT

2. $\displaystyle\int_0^1 x^{-1/2}\,dx$ (again $\displaystyle\lim_{x\to 0} x^{-1/2} = \infty$)

Taking $\displaystyle\int_0^1 x^{-1/2}\,dx = \lim_{a\to 0^+} \int_a^1 x^{-1/2}\,dx = \cancel{[illegible]}\ 2x^{1/2}\Big|_{x=1} - \overbrace{\lim_{a\to 0^+} 2x^{1/2}}^{=0} =$

$\underline{\underline{=2}}$ This integral is CONVERGENT