Differential equations

I - Definition: A differential equation is an equation involving derivatives (occurs in finance, in physics, engineering, biology, etc.)

1. Examples:

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} = F(y(t),t)$$ where

F: function of y, t

y(t): function of t

$$\frac{dy(x)}{dx} = G(y(x),x)$$ where

G: function of x, y

y: function of x

(*) Task: find y

2. Order

- Equation of the 1^{st} order: only the 1^{st} derivative is involved
- Equation of the 2^{nd} order: the 2^{nd} derivative is involved

⋮

- Equation of the n^{th} order: the n^{th} derivative is involved

Initially we will concentrate on differential equations of the 1^{st} order

see back

II - Simplest cases

$$\frac{dy}{dt} = \overbrace{2t+5}^{F(t)}$$ (here F depends only on t; i.e., F = F(t) only.)

$$\Rightarrow \int dy = \int [2t+5]\,dt \Rightarrow \boxed{y(t) = \frac{2}{2}t^2 + 5t + C}$$

where C: arbitrary constant

(*) This is called a GENERAL SOLUTION: by varying C one gets all the solutions of the above-stated equation

- Specific solutions: determined by
  - (a) Initial conditions $y(0) = C_0$ (C_0 = constant)
  - (b) Boundary conditions: $y(t_0) = C_1$ (t_0: specific value of t, C_1: constant)

• Example: (a) Determine C for $y(t) = t^2 + 5t + C$ such that $y(0) = -5$

Solution:

$$y(0) = (0)^2 + 5.(0) + C = C \quad (*)$$

On the other hand,

$$y(0) = -5 \quad (**)$$

Hence $(*) = (**) \Rightarrow C = -5$ and

$$\boxed{y(t) = t^2 + 5t - 5}$$

(b) Determine C for the same equation such that $y(1) = 0$

Solution

$$y(1) = 1 \times (1)^2 + 5(1) + C = 6 + C \quad (*)$$

$$y(1) = 0 \quad (**)$$

Then $(*) = (**) \Rightarrow 6 + C = 0 \Rightarrow C = -6$

and $$\boxed{y(t) = t^2 + 5t - 6}$$

(*) Depending on the initial / boundary conditions, we obtained different solutions from the general solution

III - Separable equations

1. Definition

Let us consider $\frac{dy}{dt} = F(y,t)$

If $F(y,t) = G(t)\,H(y)$ this is a SEPARABLE DIFFERENTIAL EQUATION

~~For instance~~

• Examples:

$$\frac{dy}{dt} = t^2 y^3 \quad \begin{cases} G(t) = t^2 \\ H(y) = y^3 \end{cases}$$

$$\frac{dy}{dt} = e^{-2t} \cos y \quad \begin{cases} G(t) = e^{-2t} \\ H(y) = \cos y \end{cases}$$

• Counter-example: $\frac{dy}{dt} = \sin(t, y^2)$

How to solve such equations?

2. Solution ("separation of variables")

- Starting point : $\frac{dy}{dt} = G(t)H(y)$

Rearranging this equation we have $\frac{1}{H(y)}\frac{dy}{dt} = G(t)$

- Second step: find a function $h(y)$ such that $\frac{dh(y)}{dy} = \frac{1}{H(y)}$

This yields $\frac{dh(y)}{dy}\frac{dy}{dt} = G(t) \Rightarrow \frac{dh(y)}{dt} = G(t)$ (*)

applying the chain rule

- Third step: integrate (*) $\Rightarrow \int dh(y) = \int G(t)\,dt$

$$\Rightarrow \boxed{\int \frac{dy}{H(y)} = \int G(t)\,dt}$$

3. Example : Solve the differential equation $\frac{dy}{dt} = \frac{e^{5t}}{2y}$

(a) Find the general solution

(b) Find the solution such that $y(0) = 1$

(a) The above-stated equation is separable : it can be written as
: $\frac{dy}{dt} = H(y)G(t)$, with $H(y) = \frac{1}{2y}$
$G(t) = e^{5t}$

- Rearranging this equation, we have
$$2y\frac{dy}{dt} = e^{5t}$$

- We consider $Y(t)$ so that $\frac{dY(t)}{dt} = 2y\frac{dy}{dt}$

- We integrate
$$\int dY = \int e^{5t}\,dt \Rightarrow \int 2y\,dy = \int e^{5t}\,dt$$

$$\Rightarrow y^2 = \frac{1}{5}e^{5t} + \underbrace{C}_{\text{arbitrary constant}}$$

Please note: even though we integrated both on the RHS and on the LHS, there exists only one arbitrary constant ~~that~~ in ~~characterized~~ a 1st order differential equation.

$$\Rightarrow \boxed{y(t) = \pm\sqrt{\frac{1}{5}e^{5t} + c}} \quad (*)$$

(b) $y(0) = 1$ ~~only the positive root satisfies this condition~~

setting $t=0$ in $(*)$ $\Rightarrow y(0) = \pm\sqrt{\frac{1}{5} + c}$

$\Rightarrow$ only the positive root satisfies this condition

$\Rightarrow 1 = \sqrt{\frac{1}{5} + c} \Rightarrow (1)^2 = \left(\sqrt{\frac{1}{5}+c}\right)^2$

$\Rightarrow C = 4/5$

Hence $\boxed{y(t) = \sqrt{\dfrac{e^{5t}+4}{5}}}$

IV - Linear equations with constant coefficients

1. Homogeneous equation: $\frac{dy}{dt} + ay = 0$, where a = const. and $a \neq 0$

How to solve this equation?

- Possibility 1: separation of variables (tell them to try at home).

- Possibility 2:

  - Differentiating a product one has $\frac{d}{dt}(u(t)v(t)) = u(t)\frac{dv(t)}{dt} + v(t)\frac{du(t)}{dt}$

  - Let us then consider the specific case

$$(*) \Rightarrow \frac{d}{dt}(y e^{at}) = \left[\frac{dy}{dt} + ay\right]e^{at} = e^{at}\frac{dy}{dt} + a e^{at} y$$

$(*)$ is zero only if $\frac{dy}{dt} + ay = 0$ (OUR DIFFERENTIAL EQUATION!!!)

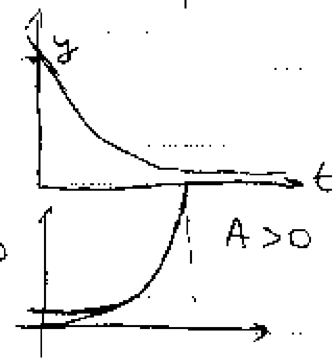
This only occurs if $y e^{at} = \text{Const} = A$

(Remember: the derivative of a constant is ZERO)

Thus $\Rightarrow y e^{at} = A \Rightarrow \boxed{y(t) = A e^{-at}}$ A determined by initial / boundary conditions

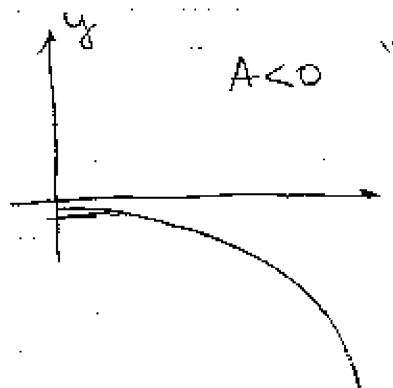
Please note:

- $a > 0 \Rightarrow y \to 0$ as $t \to \infty$
- $a < 0 \Rightarrow y \to \pm\infty$ as $t \to \infty$



"exponential decay"

A>0 "exponential growth"



A<0 "exponential growth".

Example: (a) find the general solution of

$$\frac{dy}{dt} + 5y = 0 \Rightarrow \frac{d}{dt}\left(e^{5t} y(t)\right) = 0$$

easier to see if I multiply the above equation with e^{5t}:

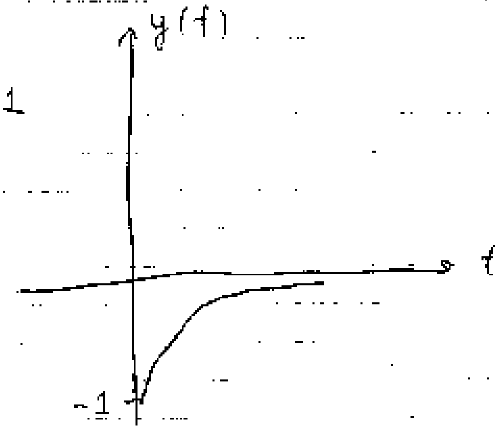
$$e^{5t}\frac{dy}{dt} + \underbrace{5y\, e^{5t}}_{\frac{d}{dt}\left[e^{5t}\right]} = 0 \Rightarrow \frac{d}{dt}\left[e^{5t}\right] \cdot y + \frac{dy}{dt} e^{5t} = 0$$

General solution: $y(t) = A e^{-5t}$

(b) find the solution such that

(i) $y(0) = 1 \Rightarrow y(0) = A \Rightarrow A = 1$

(ii) $y(0) = -1 \Rightarrow y(0) = A \Rightarrow A = -1$



2. Inhomogeneous equations

2.1. Simplest case: $\frac{dy}{dt} + ay = b$ (*) $a \neq 0$ and const. $b \neq 0$ and const.

- Case $b = 0$ solved in the previous section

How to solve this equation?

(a) Find a particular solution y_p

Guess: $y_p = C$ (C - const.)

Inserting this in the above equation $\Rightarrow \underbrace{\frac{dC}{dt}}_{=0} + aC = b \Rightarrow C = \frac{b}{a}$

$\Rightarrow \boxed{y_p = \frac{b}{a}}$

(b) Find the general solution of the associated homogeneous equation (also known as the "complementary solution")

(*) The associated homogeneous equation is found by taking $z = y - c$ and writing (*) in terms of z

This yields $\frac{dy}{dt} = \frac{dz}{dt}$ (*) reads $\frac{dz}{dt} + a(z+c) = b$

$y = z + c$

$$\frac{dz}{dt} + az + \cancel{ac} = \cancel{b}$$

$$\frac{dz}{dt} + az = 0$$

- General solution $z = Ae^{-at}$

(c) Since $y = z + c$, the general solution of our equation is $\boxed{y = \underbrace{Ae^{-at}}_{\text{solution of the associated homogeneous eq.}} + \underbrace{\frac{b}{a}}_{\text{particular solution}}}$

Example : Solve $\frac{dy}{dt} + 2y = 8$

- Step 1 : Find the particular solution : $y_p = C$

$$\underbrace{\frac{dy_p}{dt}}_{=0} + 2y_p = 8 \Rightarrow y_p = 4$$

- Step 2 : find the general solution of the associated homogeneous equation

Homogeneous differential equation:

$$\frac{dz}{dt} + 2z = 0 \quad (z = y - 4)$$

$$\Rightarrow z = Ae^{-2t}$$

- Step 3 y = particular solution + general solution of associated hom. equation

$$\boxed{y(t) = Ae^{-2t} + 4}$$

2.2 $\frac{dy}{dt} + ay = g(t)$ (constant on the RHS has been replaced by a function)

- Previous section: we found a particular solution y_p and the general solution of the associated homogeneous eq.

$y = y_p$ + General solution of the asc. h. equation

- Problem: how to find the particular solution?

Method (works for most — but not all — cases) : try a particular solution of the same form as in the RHS

Example : solve $\frac{dy}{dt} + 2y = 4e^{3t}$

Guess: $y_p = Be^{3t} \Rightarrow \frac{dy_p}{dt} = 3Be^{3t} \Rightarrow \frac{dy_p}{dt} + 2y_p = 3Be^{3t} + 2Be^{3t}$

$$\Rightarrow 5Be^{3t} = 4e^{3t} \Rightarrow B = \frac{4}{5} \text{ and } \boxed{y_p = \frac{4}{5}e^{3t}}$$

Associated homogeneous eq

$$\frac{dz}{dt} + 2z = 0 \quad (z = y - y_p) \Rightarrow z = Ae^{-2t}$$

Hence $\boxed{y = Ae^{-2t} + \frac{4}{5}e^{3t}}$

(*) Boundary conditions: solve the above equation so that $y(0) = 0$

$$y(t) = Ae^{-2t} + \frac{4}{5}e^{3t}$$

$$y(0) = A + \frac{4}{5} = 0 \Rightarrow A = -\frac{4}{5}$$

Thus $\boxed{y(t) = +\frac{4}{5}\left(e^{3t} - e^{-2t}\right)}$

(found in the same way as before)

3. Application: price adjustment

Assumption: price increases when there is excess demand and decreases when there is excess suply.

(a) Demand: $q_D = a_0 - a_1 p$ ($p \equiv$ price; a_0, a_1, b_0, b_1 constants)

(b) Suppy: $q_S = b_0 + b_1 p$

(c) Price: $\frac{dp}{dt} = \lambda(q_D - q_S)$ $(\lambda > 0)$

If the system is in equilibrium $q_D = q_S \Rightarrow a_0 - a_1 p^* = b_0 + b_1 p^*$

$$\Rightarrow p^* = \frac{a_0 - b_0}{(a_1 + b_1)}$$

Substituting (a) and (b) in (c):

$$\frac{dp}{dt} = \lambda\left[a_0 - a_1 p - (b_0 + b_1 p)\right] = -\lambda(a_1 + b_1)p + \lambda(a_0 - b_0)$$
$$+ \lambda(a_1 + b_1)\left[\frac{a_0 - b_0}{a_1 + b_1} - p\right]$$

$$\Rightarrow \frac{dp}{dt} = \lambda(a_1 + b_1)(p^* - p)$$

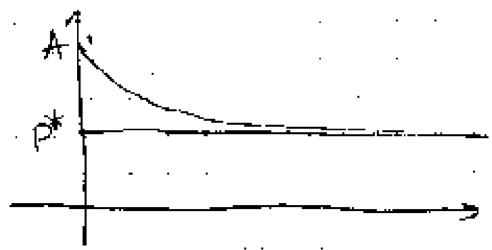
calling $\lambda(a_1 + b_1) = c \Rightarrow \frac{dp}{dt} + cp = cp^*$

Particular solution: $p = p^*$

Complementary solution: Ae^{-ct}

$p = p^* + Ae^{-ct}$, with $c = \lambda(a_1 + b_1)$

Note that: ~~∞ ...~~ $\lim_{t \to \infty} p = p^*$ (the model is stable)



- the ~~...~~ larger λ, the faster the system approaches equilibrium
- p^* is a stationary solution

V. Harder first-order equations

1. General case

$$\frac{dy}{dt} + f(t)y = g(t) \quad , f(t), g(t) \text{ functions}$$

Solution: the above equation is multiplied by an integrating factor, so that it becomes easy to integrate

Hence,

$$\Rightarrow h(t)\frac{dy}{dt} + h(t) f(t) y = h(t)\, g(t) \qquad (h(t) = \text{integrating factor})$$

if $h(t) f(t) = h'(t)$ then

$$\underbrace{h(t)\frac{dy(t)}{dt} + h'(t)\, y(t)}_{\frac{d}{dt}\left[y(t)\, h(t)\right]} = h(t)\, g(t)$$

$$\frac{d}{dt}\left[y(t)\, h(t)\right] = h(t)\, g(t)$$

So that $\quad y(t) h(t) = \int h(t)\, g(t)\, dt$

$$\boxed{y(t) = \frac{1}{h(t)}\int h(t)\, g(t)\, dt}$$

Example: Find the solution of the differential equation

$$\frac{dy}{dt} - \frac{2y}{t} = 4t$$

which satisfies the boundary condition that $y = 2$ when $t = 1$

- Step 1: find an integrating factor

we need $h(t)$ so that $\quad h'(t) = h(t)\underbrace{f(t)}_{-\frac{2}{t}}$

$$h'(t) = -\frac{2}{t}\, h(t)$$

$$\frac{dh(t)}{dt} = -\frac{2}{t}\, h(t)$$

Separation of variables: $\displaystyle\int \frac{dh(t)}{h(t)} = -2\int \frac{dt}{t}$

$$\ln h(t) = -2\ln t = \ln\left(\frac{1}{t^2}\right) \Rightarrow h(t) = \frac{1}{t^2}$$

$$\underbrace{\frac{1}{t^2}\frac{dy}{dt} - \frac{2y}{t^3}}_{\frac{d}{dt}\left(\frac{y}{t^2}\right)} = \frac{4}{t} \Rightarrow \frac{d}{dt}\left(\frac{y}{t^2}\right) = \frac{4}{t}$$

$$\frac{y}{t^2} = \int \frac{4}{t}\,dt = 4\ln t + c$$

$$\Rightarrow \boxed{y = t^2[4\ln t + c]}$$

Boundary condition : $y(1) = 2$

$$y(1) = 1^2[4\underbrace{\ln 1}_{0} + c] \Rightarrow 2 = c \quad \text{and } y = t^2[4\ln t + 2]$$

$$\Rightarrow \boxed{y = 2t^2[2\ln t + 1]}$$

1. Method for finding an integrating factor

$\frac{dh(t)}{dt} = f(t)\,h(t)$ can be written as

$$\frac{1}{h(t)}\frac{dh(t)}{dt} = f(t) \Rightarrow \frac{d}{dt}\left(\ln h(t)\right) = f(t)$$

Then $\ln h(t) = \int f(t)\,dt \Rightarrow h(t) = e^{\int f(t)\,dt}$

(In practice we find a primitive $P(t)$ of $f(t)$ and use $e^{P(t)}$ as the integrating factor).

2. Bernoulli's equation (named after Jacob Bernoulli *1654 †1705)

$$\frac{dy}{dt} + f(t)y = g(t)y^{\alpha} \quad (\alpha \neq 0 \text{ and } \alpha \neq 1)$$

Solution

$$x = y^{1-\alpha}$$

$$\frac{dx}{dt} = (1-\alpha)\,y^{-\alpha}\frac{dy}{dt}$$

Since $\frac{dy}{dt} = g(t)y^{\alpha} - f(t)y$,

$$\frac{dx}{dt} = (1-\alpha)\,y^{-\alpha}\left(g(t)y^{\alpha} - f(t)y\right) = (1-\alpha)\left(g(t) - f(t)\,y^{1-\alpha}\right)$$

$$\frac{dx}{dt} + \underbrace{(1-\alpha)}\, f(t)\, x = (1-\alpha)\, g(t)$$

(this is a harder equation; one can now solve this equation in x and transform back to y).

- Integrating factor: found solving

$$h'(t) = (1-\alpha) f(t) h(t)$$

$$\int \frac{dh(t)}{h(t)} = (1-\alpha) \int f(t)\, dt$$

$$\ln [h(t)] = (1-\alpha) \int f(t)\, dt$$

$$h(t) = \exp\left[(1-\alpha) \int f(t)\, dt\right]$$

(Procedure as in the beginning of the section)

<u>Example</u>: Find the general solution of

$$\frac{dy}{dt} - \frac{2y}{t} = 4ty^2$$

$\Rightarrow$ this is a Bernoulli equation with $f(t) = -\frac{2}{t}$

$g(t) = 4t$

$\alpha = 2$

As demonstrated, this can be reduced to

$$\frac{dx}{dt} + \underbrace{(1-\alpha)}_{-1} f(t)\, x = (1-\alpha)\, g(t)$$

$$\Rightarrow \frac{dx}{dt} - 1 \cdot \frac{-2}{t} x = -4t$$

$$\frac{dx}{dt} + \frac{2}{t} x = -4t$$

- Finding the integrating factor

$$\frac{d}{dt}\left(\ln h(t)\right) = \frac{2}{t} \Rightarrow \ln h(t) = \underbrace{\int \frac{2\, dt}{t}}_{2 \ln t = \ln(t^2)}$$

$$\Rightarrow h(t) = e^{\ln(t^2)} = t^2$$

$$\Rightarrow t^2\left[\frac{dx}{dt} + 2x\right] = \frac{d}{dt}\left[t^2 x\right] = -4t^3$$

$$t^2 x = -\underbrace{\int 4t^3\, dt} = -t^4 + c$$

$x(t) = \dfrac{c - t^4}{t^2}$ since $x = y^{1-\alpha} = y^{-1}$, $\boxed{y(t) = \dfrac{t^2}{c - t^4}}$

The general solution of $\frac{dy}{dt} + ay = f(t)$ (*) is

$y = y_p + z$, where y_p is a particular solution of (*)

and z satisfies $\frac{dz}{dt} + az = 0$

Proof:

y is a solution of (*). Thus $\frac{dy}{dt} + ay = f(t)$

Since $y = y_p + z$ we have

$$\frac{dy}{dt} = \frac{dy_p}{dt} + \frac{dz}{dt} \Rightarrow \frac{dy_p}{dt} + \frac{dz}{dt} + ay_p + az = f(t) \qquad (**)$$

Since, however, $\frac{dy_p}{dt} + ay_p = f(t)$ (this has to hold since

y_p is a particular solution of (*)), (**) yields

$$\frac{dz}{dt} + az = 0$$