(1) Homogeneous function: $f(\lambda x, \lambda y, \lambda z) = \lambda^r f(x,y,z)$

(a) $f(x,y,z) = \sqrt{\dfrac{3x^2/z + z\sqrt{x/y}}{x^3 + 3xyz + y^2x}}$

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$$f(\lambda x, \lambda y, \lambda z) = \sqrt{\dfrac{3\lambda^2x^2/(\lambda z) + \lambda z\sqrt{\lambda x/(\lambda y)}}{\lambda^3x^3 + 3\lambda^3xyz + \lambda^3y^2x}}$$

$$= \lambda^{-1} f(x,y,z)$$

$\Rightarrow$ The function is homogeneous of degree -1.

(b) $f(x,y,z) = e^{x/y} \ln\left(\dfrac{2x^3}{xyz}\right)(x^2y + yz^2 + x^2z^2)$

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$$f(\lambda x, \lambda y, \lambda z) = e^{\lambda x/\lambda y} \ln\left(\dfrac{2\lambda^3x^3}{\lambda^3xyz}\right)(\lambda^3x^2y + \lambda^3xz^2 + \lambda^4x^2z^2)$$

$$= e^{x/y} \ln\left(\dfrac{2x^3}{xyz}\right)\underbrace{(\lambda^3x^2y + \lambda^3xz^2 + \lambda^4x^2z^2)}_{(*)}$$

$\Rightarrow$ The function is not homogeneous, since the last term in (*) makes the homogeneity condition break down.

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(2)

(2) Differential equation: $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = (1-5x)e^{-2x}$

General solution:

$y = y_p + z$

y_p: Particular solution

z: complementary solution

[4]

• Complementary solution: z satisfies $\frac{d^2z}{dx^2} + 2\frac{dz}{dx} + 5z = 0$

$z = e^{\alpha x} \Rightarrow \underbrace{(\alpha^2 + 2\alpha + 5)}_{\text{characteristic equation}} e^{\alpha x} = 0$

Roots: $\alpha = -1 \pm \frac{1}{2}\sqrt{\underbrace{4-20}_{-16}}$

$\alpha = -1 \pm 2i$

$z = e^{-x}\left[Ae^{2ix} + Be^{-2ix}\right]$

• Particular solution:

$y_p = (ax+b)e^{-2x}$

[10]

$\frac{dy_p}{dx} = ae^{-2x} + 2(ax+b)e^{-2x}$

$\frac{d^2y_p}{dx^2} = -2ae^{-2x} - 2ae^{-2x} + 4(ax+b)e^{-2x}$

$\frac{d^2y_p}{dx^2} + 2\frac{dy_p}{dx} + 5y_p = -4ae^{-2x} + \cancel{4(ax+b)e^{-2x}} + 2ae^{-2x} + \cancel{4(ax+b)e^{-2x}}$
$+ 5(ax+b)e^{-2x}$

$(-4a + 2a + 5b)e^{-2x} + 5axe^{-2x} = (1-5x)e^{-2x}$

$-2a + 5b = 1$

$5a = -5 \Rightarrow a = -1$

$b = -\frac{1}{5}$

(3)

[1] $$\boxed{y = e^{-x}\left[Ae^{2ix} + Be^{-2ix}\right] - \left(\frac{1}{5}+x\right)e^{-2x}}$$

Initial conditions

[15] $$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{3}{5} \; ; \; y(0) = \frac{4}{5}$$

$$\frac{dy}{dx} = (-1+2i)A\,e^{(-1+2i)x} + (-1-2i)Be^{-2ix} + \cancel{2}\left(\frac{1}{5}+x\right)e^{-2x} - e^{-2x}$$

$$\left.\frac{dy}{dx}\right|_{x=0} = (-1+2i)A + (-1-2i)B + \underbrace{\frac{2}{5} - 1}_{-\frac{3}{5}} = -\frac{3}{5}$$

$$\left.\frac{dy}{dx}\right|_{x=0} \Rightarrow (-1+2i)A = (1+2i)B$$

$$A = \frac{(1+2i)}{-1+2i}B \qquad (*)$$

$$y(0) = A + B - \frac{1}{5} = \frac{4}{5} \Rightarrow A + B = 1$$

Inserting $(*) \Rightarrow \dfrac{1+2i-1+2i}{-1+2i}B = 1 \Rightarrow \boxed{B = \dfrac{-1+2i}{4i}}$

$$\boxed{A = \frac{1+2i}{4i}}$$

$$y = e^{-x}\left[\frac{(1+2i)}{4i}e^{2ix} + \frac{(-1+2i)}{4i}e^{-2ix}\right] - \left(\frac{1}{5}+x\right)e^{-2x}$$

$$= e^{-x}\left[\frac{1}{4i}\left(e^{2ix} - e^{-2ix}\right) + \frac{1}{2}\left(e^{2ix} + e^{-2ix}\right)\right] - \left(\frac{1}{5}+x\right)e^{-2x}$$

$$\boxed{y = e^{-x}\left(\frac{1}{2}\sin(2x) + \cos(2x)\right) - \left(\frac{1}{5}+x\right)e^{-2x}}$$

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③ ④

$$f(x,y,z) = \cos(x^2y)(x+2e^{yz}+1)$$

- Gradient:

$$Df = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \partial f/\partial z \end{pmatrix}$$, with

$\boxed{5}$

$$\frac{\partial f}{\partial x} = \cos(x^2y) - 2xy(1+2e^{yz}+x)\sin[x^2y]$$

$$\frac{\partial f}{\partial y} = 2e^{yz}z\cos(x^2y) - x^2(1+2e^{yz}+x)\sin[x^2y]$$

$$\frac{\partial f}{\partial z} = 2e^{yz}y\cos(x^2y)$$

Hessian Matrix:

$$D^2f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial x\partial z} \\ \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y\partial z} \\ \frac{\partial^2 f}{\partial z\partial x} & \frac{\partial^2 f}{\partial z\partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}$$, with

$\boxed{10}$

$$\frac{\partial^2 f}{\partial x^2} = -4x^2y^2(1+2e^{yz}+x)\cos[x^2y] - 2(1+2e^{yz}+3x)y\sin[x^2y]$$

$$\frac{\partial^2 f}{\partial y^2} = \left[x^4(1+2e^{yz}+x) + 2e^{yz}z^2\right]\cos[x^2y] - 4e^{yz}x^2z\sin[x^2y]$$

$$\frac{\partial^2 f}{\partial z^2} = 2e^{yz}y^2\cos[x^2y]$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} = x\left(-2x^2(1+2e^{yz}+x)y\cos[x^2y] - (2+3x+4e^{yz}(1+yz))\cdot \sin[x^2y]\right)$$

$$\frac{\partial^2 f}{\partial x\partial z} = \frac{\partial^2 f}{\partial z\partial x} = -4e^{yz}xy^2\sin(x^2y)$$

$$\frac{\partial^2 f}{\partial y\partial z} = \frac{\partial^2 f}{\partial z\partial y} = 2e^{yz}(1+yz)\cos(x^2y) - 2e^{yz}x^2y\sin(x^2y)$$

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx} = \cos(2x^3) - 4x^2(1 + 2e^{2x^4} + x)\sin(2x^3) +$$

$$+\left(2e^{2x^4}x^3\cos(2x^3) - x^2(1+2e^{2x^4}+x)\sin(2x^3)\right)2 + 3x^2 \cdot 2e^{2x^4} \cdot 2x \cdot$$

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$$\cos(2x^3) = \boxed{(1 + 4x^3e^{2x^4} + 12x^3e^{2x^4})\cos(2x^3) - 6x^2(1+2e^{2x^4}+x) \cdot \sin(2x^3)}$$

Implicit relations: $\alpha \ln x + c\beta \ln y + \gamma \ln z = 0 \quad (F_1)$

$$x^\alpha y^\beta z^\gamma - 5 = 0 \quad (F_2)$$

(a) Jacobian matrix:

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$$J = \begin{pmatrix} \partial F_1/\partial x & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}$$

$$\frac{\partial F_1}{\partial x} = \frac{\alpha}{x} \; ; \; \frac{\partial F_1}{\partial y} = \frac{c\beta}{y}$$

$$\frac{\partial F_2}{\partial x} = \alpha x^{\alpha-1} y^\beta z^\gamma$$

$$\frac{\partial F_2}{\partial y} = \beta x^\alpha y^{\beta-1} z^\gamma$$

$$\boxed{J = \begin{pmatrix} \frac{\alpha}{x} & \frac{c\beta}{y} \\ \alpha x^{\alpha-1} y^\beta z^\gamma & \beta x^\alpha y^{\beta-1} z^\gamma \end{pmatrix}}$$

(b) $\frac{dx}{dz}, \frac{dy}{dz} \Rightarrow \begin{bmatrix} dx/dz \\ dy/dz \end{bmatrix} = -J^{-1} \begin{bmatrix} \partial F_1/\partial z \\ \partial F_2/\partial z \end{bmatrix}$

$$\frac{\partial F_1}{\partial z} = \frac{\gamma}{z}$$

$$\frac{\partial F_2}{\partial z} = \gamma z^{\gamma-1} x^\alpha y^\beta$$

$$J^{-1} = \frac{1}{\det J}\begin{bmatrix} \partial F_2/\partial y & -\partial F_1/\partial y \\ -\partial F_2/\partial x & \partial F_1/\partial x \end{bmatrix}$$

$$\text{Det } J = \alpha\beta x^{\alpha-1} y^{\beta-1} z^{\gamma} - c\beta\alpha x^{\alpha-1} y^{\beta-1} z^{\gamma} = (1-c)\alpha\beta x^{\alpha-1} y^{\beta-1} z^{\gamma}$$

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$$J^{-1} = \frac{1}{(1-c)\alpha\beta x^{\alpha-1} y^{\beta-1} z^{\gamma}}\begin{bmatrix} \beta x^{\alpha} y^{\beta-1} z^{\gamma} & -\frac{c\beta}{y} \\ -\alpha x^{\alpha-1} y^{\beta} z^{\gamma} & \frac{\alpha}{x} \end{bmatrix}$$

$$-J^{-1}\begin{bmatrix} \frac{\gamma}{z} \\ \gamma z^{\gamma-1} x^{\alpha} y^{\beta} \end{bmatrix} = \frac{1}{(1-c)\alpha\beta x^{\alpha-1} y^{\beta-1} z^{\gamma}}\begin{bmatrix} \beta\gamma x^{\alpha} y^{\beta-1} z^{\gamma-1} - c\beta\gamma z^{\gamma-1} y^{\beta-1} x^{\alpha} \\ -\gamma\alpha x^{\alpha-1} y^{\beta} z^{\gamma-1} + \alpha\gamma x^{\alpha-1} z^{\gamma-1} y^{\beta} \end{bmatrix}$$

$$= \frac{-1}{(1-c)\alpha\beta x^{\alpha-1} y^{\beta-1} z^{\gamma}}\begin{bmatrix} (1-c)\beta\gamma x^{\alpha} y^{\beta-1} z^{\gamma-1} \\ 0 \end{bmatrix}$$

$$\boxed{\frac{dx}{dz} = -\frac{\gamma}{\alpha}\frac{x}{z}} \; ; \; \boxed{\frac{dy}{dz} = 0}$$

(c) In case $c = 1$, the Jacobian matrix is
3 not invertible, since Det $J = 0$ and therefore the derivatives are not defined.

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$$f(x,y) = 1 - x(x-\alpha)^2 - y(y+2\alpha) + 2xy$$

• Gradient:

$\boxed{3}$

$$Df = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -3x^2 + 2y + 4x\alpha - \alpha^2 \\ 2x - 2y - 2\alpha \end{pmatrix}$$

• Hessian matrix:

$\boxed{4}$ $\boxed{3}$

$$D^2 f = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} = \begin{pmatrix} -2x - 4x + 4\alpha & 2 \\ 2 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -6x + 4\alpha & 2 \\ 2 & -2 \end{pmatrix}$$

Critical points:

$\boxed{1}$

$$Df = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x - 2y - 2\alpha = 0 & (1) \\ -3x^2 + 2y + 4x\alpha - \alpha^2 = 0 & (2) \end{cases}$$

$(1) \Rightarrow x = y + \alpha$

$(2) \Rightarrow -3(y+\alpha)^2 + 2y + 4\alpha(y+\alpha) - \alpha^2 = 0$

$-3(y^2 + \alpha^2 + 2\alpha y) + 2y + 4\alpha y + \underbrace{4\alpha^2 - \alpha^2}_{3\alpha^2} = 0$

$-3y^2 \cancel{-3\alpha^2} - 6\alpha y + 2y + 4\alpha y \cancel{+3\alpha^2} = 0$

$\boxed{6}$

$-3y^2 - 2\alpha y + 2y = 0$

$\boxed{y = 0}$ or $-3y + 2 - 2\alpha = 0 \Rightarrow \boxed{y = \frac{2 - 2\alpha}{3}}$

Applying (1): $P_1 = (\alpha, 0, 1)$

$P_2 = \left(\frac{2+\alpha}{3}, \frac{2-2\alpha}{3}, \frac{1}{27}(31 - 12\alpha + 12\alpha^2 - 4\alpha^3) \right)$

Hessian matrix:

- @ P_1 : $D^2 f = \begin{pmatrix} -2\alpha & 2 \\ 2 & -2 \end{pmatrix}$ $\text{Det}[D^2 f] = 4\alpha - 4$

  $\text{Det}[D^2 f] > 0 \Rightarrow \alpha > 1$

$\boxed{1}$

- @ P_2 : $D^2 f = \begin{pmatrix} 4+2\alpha & 2 \\ 2 & -2 \end{pmatrix}$ $\text{Det}[D^2 f] = 4 - 4\alpha$

  $\text{Det}[D^2 f] > 0 \Rightarrow \alpha < 1$

(a) $\alpha = 3$

P_1 : $D^2 f = \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix}$ $\frac{\partial^2 f}{\partial x^2} < 0$, $\frac{\partial^2 f}{\partial y^2} < 0$

$\text{Det}[D^2 f] > 0 \Rightarrow$ NEGATIVE DEFINITE

P_1 is a <u>maximum</u>

$\boxed{3}$

P_2 : $D^2 f = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$ $\text{Det}[D^2 f] < 0$

$\frac{\partial^2 f}{\partial x^2} > 0$, $\frac{\partial^2 f}{\partial y^2} < 0$

$\Rightarrow P_2$ is a <u>saddle point</u>

(b) $\alpha = -1$ P_1 : $D^2 f = \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix}$ $\text{Det}[D^2 f] < 0$

$\frac{\partial^2 f}{\partial x^2} > 0$, $\frac{\partial^2 f}{\partial y^2} < 0$

$\Rightarrow P_1$ is a <u>saddle point</u>

$\boxed{3}$

P_2 : $D^2 f = \begin{pmatrix} -6 & 2 \\ 2 & -2 \end{pmatrix}$ $\text{Det}[D^2 f] < 0$

$\frac{\partial^2 f}{\partial x^2} < 0$, $\frac{\partial^2 f}{\partial y^2} < 0$

$\Rightarrow$ NEGATIVE DEFINITE

P_2 is a <u>maximum</u>!

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