4) $\frac{dy}{dt} + 2y = (3t+1)e^{-t} + e^{-2t}$

Solution: $y = \underbrace{y_p}_{\text{particular solution}} + \underbrace{z}_{\text{solution of the associated homogeneous equation (complementary solution)}}$

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- Complementary solution: $\frac{dz}{dt} + 2z = 0 \Rightarrow z = Ae^{-2t}$

- Particular solution: $y_p = y_{p_1} + y_{p_2}$ with (Hint 1)

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$$y_{p_1} = (Bt + c)e^{-t}$$

$$y_{p_2} = Dte^{-2t} \quad \text{(Hint 2)}$$

$$\frac{dy_p}{dt} = De^{-2t} - Dte^{-2t} + Be^{-t} - (Bt+c)e^{-t}$$

$$\frac{dy_p}{dt} + 2y_p = De^{-2t} + 2(Bt+c)e^{-t} + Be^{-t} - (Bt+c)e^{-t}$$

$$\frac{dy_p}{dt} + 2y_p = De^{-2t} + (Bt + C + B)e^{-t} = (3t+1)e^{-t} + e^{-2t} \quad \text{(RHS)}$$

$D = 1$

$B = 3$

$C + B = 1;\ C = -2$

$\Rightarrow y_p = (3t-2)e^{-t} + te^{-t}$

- General solution: $\boxed{y = (3t-2)e^{-t} + (t+A)e^{-2t}}$ ✓

- Specific solution:

2

$$-2 + A = 4$$

$$A = 6$$

$$\boxed{y = (3t-2)e^{-t} + (t+6)e^{-2t}}$$ ✓

2 $\dfrac{dy}{dt} - \dfrac{4t}{(1+t^2)} y = -\dfrac{2(1+t^2)}{t^3} \qquad (*)$

Integrating factor: $\dfrac{dh}{dt} = -\dfrac{4t}{(1+t^2)} h$

$$\int \frac{dh}{h} = -\int \frac{4t}{1+t^2} dt$$

5

$$\ln h = -2 \ln(1+t^2) \Rightarrow \ln h = \ln\left[\frac{1}{(1+t^2)^2}\right]$$

$$\boxed{h = \frac{1}{(1+t^2)^2}}$$

Multiplying (*) with the integrating factor:

$$\underbrace{\frac{1}{(1+t^2)^2}\frac{dy}{dt} - \frac{4t}{(1+t^2)^3} y} = -\frac{2}{t^3(1+t^2)}$$

5

$$\frac{d}{dt}\left[\frac{1}{(1+t^2)^2} y\right] = -\frac{2}{t^3(1+t^2)}$$

$$\frac{1}{(1+t^2)^2} y = -2 \underbrace{\int \frac{dt}{t^3(1+t^2)}}_{I}$$

Substitution: $t = \tan u \Rightarrow dt = \sec^2 u \, du$

$1+t^2 = \sec^2 u$

$t^3 = \tan^3 u$

$$I = -2\int \frac{du}{\tan^3 u} = -2\int \frac{\cos^3 u}{\sin^3 u} du = -2\int \frac{(1-\sin^2 u)\cos u \, du}{\cos^3 u}$$

$$-2\int \frac{\cos u}{\sin^3 u} du = \frac{\cancel{-2}}{\cancel{-2}} (\sin u)^{-2} = \frac{1}{\sin^2 u} \qquad (**)$$

$$2\int \frac{\sin^2 u \cos u}{\sin^3 u} du = 2 \ln(\sin u) \qquad (***)$$

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$$(**) + (***) = \frac{1}{\sin^2 u} + 2\ln(\sin u) + c$$

$\tan u = t$

$$\frac{1}{\cos^2 u} = 1+t^2 \Rightarrow \cos t = \frac{1}{\sqrt{1+t^2}} \Rightarrow \sin u = \tan u \cos u$$

$$\sin u = \frac{t}{\sqrt{1+t^2}}$$

$$(**) + (***) = \frac{1+t^2}{t^2} + 2\ln\left[\frac{t}{\sqrt{1+t^2}}\right] + c$$

$$\frac{1}{(1+t^2)^2}\, y = \frac{1+t^2}{t^2} + 2\ln\left[\frac{t}{\sqrt{1+t^2}}\right] + c$$

$$\boxed{y = \frac{(1+t^2)^3}{t^2} + 2(1+t^2)^2 \ln\left[\frac{t}{\sqrt{1+t^2}}\right] + c(1+t^2)^2}$$

Specific solution: $y(1) = 4 - 4\ln 2$

$$y(1) = 8 + \underbrace{8\ln\left[\frac{1}{\sqrt{2}}\right]}_{-4\ln 2} + 4c = 4 - 4\ln 2$$

2

$$\Rightarrow 8 - \cancel{4\ln 2} + 4c = 4 - \cancel{4\ln 2}$$

$$4c = -4$$

$$\boxed{c = -1}$$

$$\boxed{y = \frac{(1+t^2)^3}{t^2} + 2(1+t^2)^2 \ln\left[\frac{t}{\sqrt{1+t^2}}\right] - (1+t^2)^2}$$

Please note: This result is equivalent to

$y = (1+t^2)^2\left[\frac{1}{t^2} + \ln\left[\frac{t}{1+t^2}\right]\right]$ (found using partial fractions and having $c=0$)

3) $\dfrac{dy}{dt} + \dfrac{1}{(1-t^2)} y = (1+t^2) y^3$ (Bernoulli equation)

Change of variables: $x = y^{-2}$

$$\frac{dx}{dt} = -2y^{-3}\frac{dy}{dt} = -2y^{-3}\left[(1+t^2)y^3 - \frac{1}{(1-t^2)}y\right] = -2(1+t^2) + \frac{2}{(1-t^2)}x$$

$$\frac{dx}{dt} - \frac{2x}{(1-t^2)} = -2(1+t^2)$$ (can be solved using the integrating factor)

$$\frac{h'(t)}{h(t)} = -\frac{2}{(1-t^2)}$$

$$\int \frac{dh}{h} = -2\int \frac{1}{(1-t^2)}dt = -2\int \frac{1}{(1+t)(1-t)}dt$$

Partial fractions: $\dfrac{1}{(1+t)(1-t)} = \dfrac{A}{(1+t)} + \dfrac{B}{(1-t)} = \dfrac{A(1-t)+B(1+t)}{(1+t)(1-t)}$

$A = 1/2$

$B = 1/2$

$\boxed{0}$

$$-\int \frac{1}{(1+t)(1-t)}dt = -\left[\int \frac{dt}{1+t} + \int \frac{dt}{1-t}\right] = -\left[\ln(1+t) - \ln(1-t)\right]$$

$$= \ln\left[\frac{1-t}{1+t}\right] \quad (-1<t<1) \quad \Rightarrow \ln h = \ln\left[\frac{1-t}{1+t}\right]$$

$$\boxed{h = \frac{1-t}{1+t}}$$

Check: $\dfrac{d}{dt}\left[\dfrac{1-t}{1+t}\right] = -\dfrac{2}{(1-t^2)}$ ✓

$\boxed{3}$ $\dfrac{d}{dt}\left[\dfrac{1-t}{1+t}x\right] = \dfrac{-2(1+t^2)(1-t)}{1+t}$

$$\frac{1-t}{1+t} \times \quad = -2\underbrace{\int \frac{(1+t^2)(1-t)}{1+t}\,dt}_{I}$$

$$I = \int \frac{(1+(u-1)^2)\ (2-u)}{u}\,du \qquad (\text{substitution: } 1+t = u \Rightarrow du = dt,\ \ 1-t = 2-u)$$

$$= \underbrace{\int \frac{2-u}{u}\,du}_{I_1} + \underbrace{\int \frac{(2-u)(u^2-2u+1)}{u}\,du}_{I_2} = \int u(2-u)\,du$$

$$I_1 = 2\ln u - u$$

$$I_2 = \underbrace{\int u(2-u)\,du}_{u^2 - \frac{u^3}{3}} - 2\underbrace{\int (2-u)\,du}_{-4u+u^2} + \underbrace{\int \frac{2-u}{u}\,du}_{2\ln u - u}$$

$$I = 4\ln u - 2u + u^2 - \frac{u^3}{3} - 4u + u^2 = 4\ln u + 2u^2 - 6u - \frac{u^3}{3}$$

$$= 2(1+t)^2 - \frac{(1+t)^3}{3} - 6(1+t) + 4\ln(1+t)$$

$$= 2(1+t^2+2t) - \frac{(1+3t+3t^2+t^3)}{3} - 6(1+t) + 4\ln(1+t)$$

$$= \underbrace{2 - \frac{1}{3} - 6}_{-\frac{13}{3}} + \underbrace{4t - t - 6t}_{-3t} + \underbrace{2t^2 - t^2}_{t^2} - \frac{t^3}{3}$$

$$I = 4\ln[1+t] + t^2 - 3t - \frac{t^3}{3} \underbrace{- \frac{13}{3} + c}_{\text{can be combined}}$$

$$-2I = -8\ln[1+t] - 2t^2 + 6t + \frac{2t^3}{3} + \check{c}$$

$$x(t) = \frac{(1+t)}{(1-t)}\left[-8\ln[1+t] - 2t^2 + 6t + \frac{2t^3}{3} + \check{c}\right]$$

$\boxed{2}$

$$\boxed{y(t) = \pm\left[\frac{(1-t)}{(1+t)\left[-8\ln[1+t] - 2t^2 + 6t + \frac{2t^3}{3} + \check{c}\right]}\right]^{1/2}}$$

(4) $z = z_1^{30} + z_2^{100}$

$$z_1 = \left(\frac{1+i\sqrt{3}}{\sqrt{3}-i}\right), \quad z_2 = \left(\frac{1-i}{\sqrt{2}(1+i)} + \frac{1}{\sqrt{2}}\right)$$

6 $$z_1 = \frac{(1+i\sqrt{3})(\sqrt{3}+i)}{(\sqrt{3}-i)(\sqrt{3}+i)} = \frac{\cancel{\sqrt{3}}+i+3i-\cancel{\sqrt{3}}}{4} = i = e^{i\pi/2}$$

$$z_1^{30} = \left(e^{i\pi/2}\right)^{30} = e^{i30\pi/2} = e^{i15\pi} = \cos(15\pi) + i\sin(15\pi) = -1$$

8 $$z_2 = \frac{(1-i)}{\sqrt{2}(1+i)} \times \frac{(1-i)}{(1-i)} + \frac{1}{\sqrt{2}} = \frac{1-1-2i}{2\sqrt{2}} + \frac{1}{\sqrt{2}} = -\frac{i}{\sqrt{2}} + \frac{1}{\sqrt{2}}$$

Trigonometric form: $z_2 = |z_2|\, e^{i\theta_2}$, $\theta_2 = \mathrm{ArcTan}[-1] = \frac{7\pi}{4}$

$|z_2| = 1$

$$z_2^{100} = \cos\left(100 \times \frac{7\pi}{4}\right) + i\sin\left(100 \times \frac{7\pi}{4}\right) = \cos(175\pi) + i\sin(175\pi) = -1$$

$$z_1^{30} + z_2^{100} = -2$$

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5 $\int x e^{-x} \cos x \, dx = \int x e^{-x} \left[\frac{e^{ix} + e^{-ix}}{2}\right] dx = \frac{I_1 + I_2}{2}$

$I_1 = \int x e^{-(1-i)x} dx$

Integration by parts

$u = x \Rightarrow du = 1$

$dv = e^{-(1-i)x} dx \Rightarrow v = -\frac{1}{(1-i)} e^{-(1-i)x}$

$I_1 = -\frac{e^{-(1-i)x}}{(1-i)} + \frac{1}{(1-i)} \int e^{-(1-i)x} dx = -\frac{e^{-(1-i)x}}{(1-i)} - \frac{e^{-(1-i)x}}{(1-i)^2}$

$I_2 = \int x e^{-(1+i)x} dx$

Integration by parts:

$u = x \Rightarrow du = 1$

$dv = e^{-(1+i)x} dx \Rightarrow v = \frac{-1}{(1+i)} e^{-(1+i)x}$

$I_2 = -\frac{e^{-(1+i)x}}{(1+i)} + \frac{1}{(1+i)} \int e^{-(1+i)x} dx = -\frac{e^{(1+i)x}}{(1+i)} - \frac{e^{-(1+i)x}}{(1+i)^2}$

$I_1 + I_2 = -x e^{-x} \underbrace{\left[\frac{e^{ix}}{(1-i)} + \frac{e^{-ix}}{(1+i)}\right]}_{(*)} - e^{-x} \underbrace{\left[\frac{e^{ix}}{(1-i)^2} + \frac{e^{-ix}}{(1+i)^2}\right]}_{(**)}$

$$(*) \Rightarrow \frac{(1+i)e^{ix}+(1-i)e^{-ix}}{\underbrace{(1-i)(1+i)}_{2}} = \underbrace{\frac{e^{ix}+e^{-ix}}{2}}_{\cos x} + \underbrace{\frac{i}{2}\left[e^{ix}-e^{-ix}\right]}_{-\sin x}$$

$$(**) \Rightarrow \frac{(1+i)^2 e^{ix}+(1-i)^2 e^{-ix}}{\underbrace{(1-i)^2(1+i)^2}_{4}} = \frac{(\cancel{1-1}+2i)\,e^{ix}+(\cancel{1-1}-2i)\,e^{-ix}}{4} = \underbrace{\frac{i}{2}\left(e^{ix}-e^{-ix}\right)}_{-\sin x}$$

$$I = \frac{I_1+I_2}{2} = -\frac{x}{2}e^{-x}(\cos x-\sin x)+e^{-x}\frac{\sin x}{2} + C$$

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