Further mathematics for economists

Coursework 2 - Complex numbers/2^{nd}–order differential equations/functions of several variables

1. (10/100) Compute $\left(1/2 + i\sqrt{3}/2\right)^4 + (1/\sqrt{2} - i/\sqrt{2})^8$

2. (20/100) Find all the roots of $z^6 = -64$. Write these roots in algebraic form

3. (15/100) Consider the function of two variables $f(x, y) = (x^2 + e^{3y})\cos(2xy)$

   (a) Compute the gradient and the Hessian matrix of $f(x, y)$
   Hint. remember the properties of mixed derivatives of smooth functions

   (b) Compute the **total** derivative $df(x, y)/dx$ for $y = 2x$

4. (25/100) Find the specific solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^x - 5e^{2x}$$

   so that $y(0) = 0$ and $dy/dx = 1$ when $x = 0$ (Hint: use $y_p = axe^x + bxe^{2x}$)

5. (30/100) Find the specific solution of the second-order differential equation

$$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 25y = 16\sin x + 8\cos x,$$

   so that $y(0) = 1$ and $dy/dx = 0$ if $x = 0$. Write your answer in terms of trigonometric functions
   Hints: use

   - $y_p = A\cos x + B\sin x$
   - $\sin u = \frac{1}{2i}(e^{iu} - e^{-iu})$
   - $\cos u = (e^{iu} + e^{-iu})/2$

COURSEWORK 2 - Further maths for economists Solution

1. $$\underbrace{\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)}_{z_1}^4 + \underbrace{\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)}_{z_2}^8$$

$$z_1 = 1\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) = 1e^{i\pi/3} \Rightarrow z_1^4 = e^{i\frac{4\pi}{3}}$$

$\boxed{4}$ $$z_1^4 = \cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$z_2 = 1\left[\cos\left(2\pi-\frac{\pi}{4}\right)+i\sin\left(2\pi-\frac{\pi}{4}\right)\right] = e^{i7\pi/4}$$

$\boxed{6}$ $$z_2^8 = e^{i14\pi} = 1$$

10

$$z_1+z_2 = \frac{1}{2} - \frac{i\sqrt{3}}{2}$$

2. $\boxed{2}$ $$z^6 = -64 = 2^6 e^{i\pi + i2n\pi}$$

$$z = 2e^{i\pi/6} e^{\frac{i2n\pi}{6}}$$

$\boxed{3}$ $$n=0 \Rightarrow z_1 = 2e^{i\pi/6} = 2\left(\cos\frac{\pi}{6}+i\frac{\pi}{6}\right) = 2\overbrace{\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)}^{\sqrt{3}+i}$$

$\boxed{3}$ $$n=1 \Rightarrow z_2 = 2e^{i\pi/6}e^{i\pi/3} = 2\left(\cos\frac{3\pi}{6}+i\sin\frac{3\pi}{6}\right) = 2i$$

$\boxed{3}$ $$n=2 \Rightarrow z_3 = 2e^{i\pi/6}e^{i2\pi/3} = 2e^{i\frac{5\pi}{6}} = 2e^{i\left(\pi-\frac{\pi}{6}\right)} =$$

$$= 2\left(\cos\left(\pi-\frac{\pi}{6}\right)+i\sin\left(\pi-\frac{\pi}{6}\right)\right] = +2\left[-\frac{\sqrt{3}}{2}+\frac{i}{2}\right]$$

$$= -\sqrt{3}+i$$

$\boxed{3}$ $$n=3 \Rightarrow z_4 = 2e^{i\pi/6}e^{i\pi} = -2\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right) = -\sqrt{3}-i$$

(c.c. of z_3)

$\boxed{3}$ $$n=4 \Rightarrow z_5 = 2e^{i\pi/6}e^{i\frac{4\pi}{3}} = 2e^{i\pi/6}e^{i\pi}e^{i\pi/3} = -z_2$$

$= -2i$ (c.c. of z_2)

(1)

$n=5 \Rightarrow z_6 = 2e^{i\pi/6} \cdot e^{i5\pi/3} = 2\, e^{i\pi} \cdot e^{i\pi/6} \cdot e^{i2\pi/3} = -2\, e^{i\pi/6} \cdot e^{i2\pi/3}$

$\boxed{3}$ $= -z_3 = \sqrt{3} - i$ (c.c of z_1)

(20)

(3) $f(x,y) = (x^2 + e^{3y}) \cos(2xy)$

(a)

$$Df(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \\ \frac{\partial f}{\partial y} \end{pmatrix}$$

(4)

$$\frac{\partial f}{\partial x} = 2x \cos 2xy + (x^2 + e^{3y}) \cdot (-2y \sin(2xy))$$
$$= 2x \cos 2xy - 2y(x^2 + e^{3y}) \sin(2xy)$$

$$\frac{\partial f}{\partial y} = 3e^{3y} \cos(2xy) + (x^2 + e^{3y}) \cdot (-2x \sin 2xy)$$
$$= 3e^{3y} \cos(2xy) - 2x(x^2 + e^{3y}) \sin 2xy$$

(12)

$$Df(x,y) = \begin{pmatrix} 2x \cos 2xy - 2y(x^2 + e^{3y}) \sin(2xy) \\ \\ 3e^{3y} \cos 2xy - 2x(x^2 + e^{3y}) \sin(2xy) \end{pmatrix}$$

(8)

$$D^2 f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \quad \text{with}$$

$$\frac{\partial^2 f}{\partial x^2} = 2\cos 2xy - 4xy \sin(2xy) - 4y^2 \cos(2xy)(x^2 + e^{3y})$$
$$- 4xy \sin(2xy) = 2\cos 2xy - 8xy \sin 2xy - 4y^2(x^2 + e^{3y}) \times$$
$$\times \cos 2xy$$

$$\frac{\partial^2 f}{\partial y^2} = 9e^{3y} \cos(2xy) - 6x e^{3y} \sin(2xy) - 4x^2(x^2 + e^{3y}) \cos 2xy +$$
$$- 6x e^{3y} \sin 2xy$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -4xy(e^{3y}+x^2)\cos 2xy - 4x^2 \sin(2xy) + 2(e^{3y}+x^2)\sin(2xy) - 6e^{3y} y \sin(2xy)$$

(b) $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$

$\boxed{3}$ $y = 2x \Rightarrow \frac{dy}{dx} = 2$

$$\frac{df}{dx} = 2x\cos(4x^2) - 4x(x^2+e^{6x})\sin(4x^2) + \left[3e^{6x}\cos(4x^2) - 2x(x^2+e^{6x})\sin(4x^2)\right]\times 2$$

$$\Rightarrow \frac{df}{dx} = 2(x+3e^{6x})\cos 4x^2 - 8x(x^2+e^{6x})\sin 4x^2$$

15

4 $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 2e^x - 5e^{2x} \qquad (*)$$

- General solution : $y = y_p + z$ → solution of the associated homogeneous equation

  y_p → particular solution

- Find y_p :

$y_p = Axe^x + Bxe^{2x}$

$$\frac{dy_p}{dx} = Ae^x + Axe^x + Be^{2x} + 2Bxe^{2x}$$

10

$$\frac{d^2y_p}{dx^2} = Ae^x + Ae^x + Axe^x + 2Be^{2x} + 2Be^{2x} + 4Bxe^{2x}$$
$$= (2A + Ax)e^x + (4B + 4Bx)e^{2x}$$

Inserting into $(*)$:

$$2Ae^x + Axe^x + 4Be^{2x} + 4Bxe^{2x} - 3Ae^x - 3Axe^x - 3Be^{2x} - 6Bxe^{2x}$$
$$+ 2Axe^x + 2Bxe^{2x} = 2e^x - 5e^{2x}$$

$$\underbrace{(2A - 3A)}_{-A}e^x + (\cancel{3Ax} - \cancel{3Ax})e^x + \underbrace{(4B - 3B)}_{B}e^{2x} - \cancel{4Bxe^{2x}} + \cancel{4Bxe^{2x}}$$
$$= 2e^x - 5e^{2x}$$

$-A = 2$; $B = -5$ $\Rightarrow y_p = -2xe^x - 5xe^{2x}$

- Complementary solution:

$$\frac{d^2z}{dx^2} - 3\frac{dz}{dt} + 2z = 0$$

10

$z = e^{\alpha t}$

$\alpha^2 - 3\alpha + 2 = 0$ $\Rightarrow z = Ae^x + Be^{2x}$

$\alpha_1 = +\frac{3}{2} + \frac{1}{2}\sqrt{9-8} = \mathbf{2}$ $\qquad y = -2xe^x - 5xe^{2x} + Ae^x + Be^{2x}$

$\alpha_2 = \frac{3}{2} - \frac{1}{2}\sqrt{9-8} = 1$

5

• Specific solution:

$y(0) = 0 \Rightarrow A + B = 0$

$\left.\frac{dy}{dx}\right|_{x=0} = 1 \Rightarrow \frac{dy}{dx} = -2e^{x} - 2xe^{x} - 5e^{2x} - 10xe^{2x} + Ae^{x} + 2Be^{2x}$

$\left.\frac{dy}{dx}\right|_{x=0} = -2 - 5 + A + 2B = 1$

5

$A + 2B = 8$

$-B + 2B = 8 \Rightarrow B = 8$

$A = -8$

$$\boxed{y = -2xe^{x} - 5xe^{2x} + 8e^{2x} - 8e^{x}}$$

25

5 $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 25y = 16\sin x + 8\cos x \quad (*)$

$y = \underbrace{y_p}_{\text{particular solution}} + \underbrace{z}_{\text{solution of associated homogeneous equation}}$

• Particular solution: $y_p = A\sin x + B\cos x$

10

$\frac{dy_p}{dx} = A\cos x - B\sin x$

$\frac{d^2y_p}{dx^2} = -A\sin x - B\cos x$

Inserting in $(*)$

$-A\sin x - B\cos x - 8A\cos x + 8B\sin x + 25A\sin x + 25B\cos x = 16\sin x + 8\cos x$

$\Rightarrow (24A + 8B)\sin x + (24B - 8A)\cos x = 16\sin x + 8\cos x$

$24A + 8B = 16 \quad /8$

$24B - 8A = 8 \quad /8 \Rightarrow$

$3A + B = 2$

$3B - A = 1$

$B = 2 - 3A \Rightarrow 6 - 9A - A = 1 \Rightarrow A = 1/2 \; ; B = 1/2$ (6)

$y_P = \frac{1}{2}\sin x + \frac{1}{2}\cos x$

· Complementary solution:

$$\frac{d^2 z}{dx^2} - 8\frac{dz}{dx} + 25z = 0 \qquad \text{Ausatz: } z = e^{\alpha x}$$

(10)

$$\alpha^2 - 8\alpha + 25 = 0$$

$$\alpha = 4 \pm \frac{1}{2}\sqrt{\underbrace{64 - 100}_{-36}}$$

$$\alpha = 4 \pm 3i \Rightarrow z = A e^{4x + 3ix} + B e^{4x - 3ix}$$

$$y = z + y_P = A e^{(4+3i)x} + B e^{(4-3i)x} + \frac{1}{2}(\sin x + \cos x)$$

(*) Specific solution:

(10)

$$y(0) = 1 \Rightarrow A + B + \tfrac{1}{2} = 1 \Rightarrow A = -B + 1/2$$

$$\left.\frac{dy}{dx}\right|_{x=0} = 0 \qquad \frac{dy}{dx} = (4+3i) A e^{(4+3i)x} + (4-3i) B e^{4x - 3ix} + \frac{1}{2}(\cos x - \sin x)$$

$$\left.\frac{dy}{dx}\right|_{x=0} = (4+3i)A + (4-3i)B + \frac{1}{2} = 0$$

$$(4+3i)(-B + 1/2) + (4-3i)B + \frac{1}{2} = 0$$

$$-6iB + \frac{1}{2}(4 + 3i + 1) = 0$$

$$-6i B = -\frac{5}{2} - \frac{3i}{2}$$

$$B = \frac{3}{12} + \frac{5}{12i}$$

$$A = \frac{6}{12} - \frac{3}{12} - \frac{5}{12i} = \frac{3}{12} - \frac{5}{12i}$$

7

$$y = e^{4x}\left[\left(\frac{1}{4} - \frac{5}{12i}\right)e^{3ix} + \left(\frac{1}{4} + \frac{5}{12i}\right)e^{-3ix}\right] + \frac{1}{2}(\sin x + \cos x)$$

$$= e^{4x}\left[\frac{1}{2}\left(\frac{e^{3ix} + e^{-3ix}}{2}\right) - \frac{5}{6}\left(\frac{e^{3ix} - e^{-3ix}}{2i}\right)\right] + \frac{1}{2}(\sin x + \cos x)$$

$$= e^{4x}\left[\frac{1}{2}\cos 3x - \frac{5}{6}\sin 3x\right] + \frac{1}{2}(\sin x + \cos x)$$

30