Further mathematics for economists
Coursework 1 - Integration/Differential Equations[1]

1. (10/100) Find the area below the curve $y(x) = x^3 + 2x^2 + 2$ delimited by the lines $x = 1$ and $x = 2$

2. (10/100) Find the area above the parabola $f_1(x) = 3x^2$ and below the parabola $f_2(x) = 1 - x^2$

3. (10/100) Compute the indefinite integral

$$\int \frac{dx}{x^2 - 4}$$

   Hint: Note that $1/(x^2 - 4)$ can be written as $A/(x + a) + B/(x + b)$, where A, B, a, b are constants

4. (15/100) Show that $\int \arcsin x dx = x \arcsin x + \sqrt{1 - x^2} + C$
   Hints:

   - You will need to do one integration by parts and one by substitution
   - Note that $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$

5. (20/100) Find the general solution of the differential equation

$$2\frac{dy}{dt} + 4y = 8e^{2t}$$

6. (20/100) Find the specific solution of the differential equation

$$x \cos x \frac{dy}{dx} + (x \sin x + \cos x)y = \sin x, 0 < x < \pi/2$$

   so that $y(\pi/3) = 0$
   Hints:

   - You will need to employ an integrating factor
   - You will need to solve two integrals involving trigonometric functions by substitution

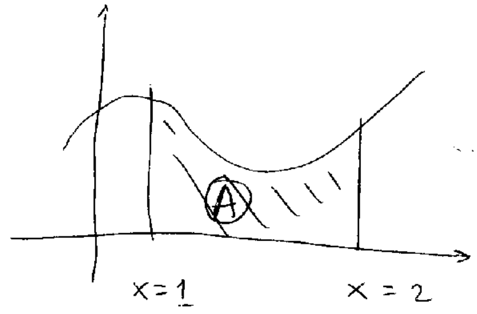
7. (15/100) Find the general solution of the Bernoulli equation

$$\frac{dy}{dt} + 4y = ty^3$$

[1] The material to the course can be found at: http://www.staff.city.ac.uk/c.f.m.faria/furthermaths.html

COURSEWORK 1 - Solutions

1) $y_1(x) = x^3 + 2x^2 + 2$



$$A = \int_1^2 (x^3 + 2x^2 + 2)\,dx = \left.\frac{x^4}{4} + \frac{2x^3}{3} + 2x\right]_1^2$$

$$A = \underbrace{\frac{(2^4)}{4}}_{4} + \frac{\overbrace{2 \times 8}^{16}}{3} + 4 - \frac{1}{4} - \frac{2}{3} - 2 =$$

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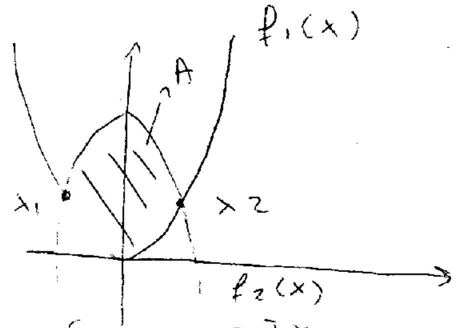
2) $f_1(x) = 3x^2$

$f_2(x) = 1 - x^2$



$\boxed{6}$ $A = \int_{x_1}^{x_2} (1 - 4x^2)\,dx = \left[x - \frac{4x^3}{3}\right]_{x_1}^{x_2}$

$\boxed{3}$. Find $x_1, x_2 \Rightarrow f_1(x) = f_2(x)$ at x_1, x_2

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$3x^2 = 1 - x^2 \Rightarrow 4x^2 = 1 \Rightarrow x = \pm \frac{1}{2}$

$\boxed{1}$ $A = \frac{1}{2} - \frac{4}{3} \cdot \frac{1}{8} + \frac{1}{2} - \frac{4}{3} \cdot \frac{1}{8} = 1 - \frac{1}{3} = \frac{2}{3}$

3) $I = \int \frac{dx}{x^2 - 4}$

$\boxed{5}$ $\frac{1}{x^2 - 4} = \frac{A}{x+2} + \frac{B}{x-2} = \frac{A(x-2) + B(x+2)}{(x^2+4)}$ < $B = \frac{1}{4}$, $A = -\frac{1}{4}$

$\boxed{5}$ $I = \int \frac{-1\,dx}{4(x+2)} + \frac{1}{4}\int \frac{dx}{(x-2)} = \frac{-1}{4}\ln|x+2| + \frac{1}{4}\ln|x-2| + c$ 10

$\Rightarrow \frac{1}{4}\ln\left[\left|\frac{x-2}{x+2}\right|\right] + C$

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[4] $\int \underbrace{\arcsin x}_{u} \, \underbrace{dx}_{dv} = x \arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}}$ $(*)$

[10] $I = \int \frac{x\,dx}{\sqrt{1-x^2}}$ By substitution:

$u = 1 - x^2$

$\frac{du}{dx} = -2x$

$du = -2x\,dx$

$I = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -\frac{1}{2} \times \frac{\sqrt{u}}{1/2} + C = -\sqrt{1-x^2} + C$

[1] Inserting into $(*)$ $\Rightarrow$ $\int \arcsin x\,dx = x \arcsin x + \sqrt{1-x^2} + C$

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$2\frac{dy}{dt} + 4y = 8e^{2t}$

$\frac{dy}{dt} + 2y = 4e^{2t}$

- Find the particular solution:

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$y_p = ae^{2t}$

$\frac{dy_p}{dt} = 2ae^{2t} \Rightarrow (2a + 2a)e^{2t} = 4e^{2t}$

$a = 1$

$y_p = e^{2t}$

- Find the complementary solution:

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$\frac{dz}{dt} + 2z = 0$ $(z = y - y_p)$

$z = Ae^{-2t}$

$$\boxed{y = e^{2t} + Ae^{-2t}}$$

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$$x\cos x\,\frac{dy}{dx} + (x\sin x + \cos x)\,y = \sin x \quad , \quad 0 < x < \pi/2$$

$$\frac{dy}{dx} + \left(\frac{\sin x}{\cos x} + \frac{1}{x}\right)y = \frac{\sin x}{x\cos x}$$

• Find the integrating factor

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$$e^{\int\left(\frac{\sin x}{\cos x} + \frac{1}{x}\right)dx} = e^{-\ln(\cos x) + \ln x} = e^{\ln(x/\cos x)} = \frac{x}{\cos x}$$

$$\frac{x}{\cos x}\frac{dy}{dx} + \frac{x}{\cos x}\left(\frac{\sin x}{\cos x} + \frac{1}{x}\right)y = \frac{\sin x}{\cos^2 x}$$

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$$\frac{d}{dx}\left(\frac{x}{\cos x}y\right) = \frac{\sin x}{\cos^2 x}$$

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$$\frac{x}{\cos x}y = \int \frac{\sin x\,dx}{\cos^2 x} = -\int \frac{du}{u^2} = \frac{1}{u} + C = \frac{1}{\cos x} + C$$

$u = \cos x$

$du = -\sin x\,dx$

$$\Rightarrow y = \frac{1}{x} + C\,\frac{\cos x}{x}$$

2 $\quad y(\pi/3) = 0$

$$\frac{1}{\pi/3} + C\,\frac{\cos\frac{\pi}{3}}{\pi/3} = 0$$

$$C = \frac{-1}{\cos \pi/3} = -2$$

$$\boxed{y = \frac{1}{x} - 2\,\frac{\cos x}{x}}$$

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7 $\frac{dy}{dt} + 4y = ty^3$

Change of variable

$x = y^{-2}$ $\Rightarrow \frac{dx}{dt} = -2y^{-3}\frac{dy}{dt} = -2y^{-3}(ty^3 - 4y)$

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$\Rightarrow \frac{dx}{dt} = -2t + 8x \quad \Rightarrow \frac{dx}{dt} - 8x = -2t$

• Particular solution: $x_p = at + b \Rightarrow \frac{dx_p}{dt} - 8x_p = -2t$

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$\frac{dx_p}{dt} = a$

$a - 8b - 8at = -2t$

$a = \frac{1}{4}$

$b = \frac{1}{32}$

• Complementary solution ($z = x - x_p$)

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$\frac{dz}{dt} - 8z = 0 \Rightarrow z = Ae^{8t}$

$x = \frac{t}{4} + Ae^{8t} + \frac{1}{32}$

$$\boxed{y = \pm \frac{1}{\sqrt{t/4 + Ae^{8t} + \frac{1}{32}}}}$$

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Possibility 2: find the integrating factor

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$f(t) = e^{-\int 8dt} = e^{-8t}$

$\frac{dx}{dt}e^{-8t} - 8e^{-8t}x = -2te^{-8t} \Rightarrow \frac{d}{dt}(e^{-8t}x) = -2te^{-8t}$

$\therefore e^{-8t}x = -2\int te^{-8t}dt$

By parts: $u = -2t$

$dv = e^{-8t}$

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$$-2\int te^{-8t}\,dt = \frac{-2}{-8}te^{-8t} + \frac{2}{-8}\int e^{-8t}\,dt = \frac{te^{-8t}}{4} + \frac{1}{32}e^{-8t} + c$$

$$\Rightarrow e^{-8t}x = \frac{t}{4}e^{-8t} + \frac{1}{32}e^{-8t} + c$$

$$x = \frac{t}{4} + \frac{1}{32} + Ce^{+8t}$$

$$\boxed{y = x^{-1/2} = \pm\frac{1}{\sqrt{t/4 + \frac{1}{32} + ce^{8t}}}}$$

$\boxed{15}$