Complex numbers

I. Generalities

1. Definition

Let us consider a number i such that $i^2 = -1$. A complex number $\in \mathbb{C}$ is written as

$$z = \underbrace{a}_{\substack{\text{real}\\\text{part}\\Re[z]=a}} + \underbrace{bi}_{\substack{\text{Imaginary part}\\Im[z]=b}}, \text{ where } a, b \text{ are real numbers}$$

- If $Im[z] = 0$, z is Real ($\mathbb{R}$ is subset of $\mathbb{C}$)
- If $Re[z] = 0$, z is imaginary (

(*) Complex numbers obey the usual algebraic laws of addition, multiplication and division, remembering that $i^2 = -1$

2. Conjugates

Consider the complex number $z = a + bi$. Its conjugate is given by $z^* = a - bi$ $\Rightarrow$ $Im[z] = -Im[z^*]$

$\bar{z} = a - bi$ $\qquad Re[z] = Re[z^*]$

3. Modulus

If z is a complex number with real part a and imaginary part b; then $|z| = \sqrt{a^2+b^2}$ (this is a generalization to $\mathbb{C}$ of the absolute value; $|z| > 0$ always holds and if $b = 0$, $|z| = |a|$)

(*) Please note: $|z| = \sqrt{z z^*}$

Proof: $z = a + bi$
$z^* = a - bi$ $\Rightarrow z z^* = (a+bi)(a-bi) = a^2 - \underbrace{(ib)^2}_{i^2 \times b^2}$

Since $i^2 = -1$, $z z^* = a^2 + b^2 = |z|^2$

4. Multiplication

Let us consider $z_1 = a + bi$ $\quad (a, b, c, d, \text{ real})$

$z_2 = c + di$

The product $z_1 z_2$ is $(a+bi)(c+di) = (ac - bd) + i(ad + bc)$

Proof: $(a+bi)(c+di) = ac + iad + ibc + \underbrace{i^2}_{-1} bd =$

$= (ac - bd) + i(ad+bc) = z_3$

Analogously $z_1^* z_2^* = (ac-bd) - i(ad+bc) = z_3^*$

Hence

$z_3^* z_3 = z_1^* z_2^* z_1 z_2 = z_1^* z_1 z_2^* z_2$

$|z_3|^2 = |z_1|^2 |z_2|^2 \Rightarrow |z_3| = |z_1| |z_2|$

5. Division

If z_1 and z_2 are complex numbers, with $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{z_1 z_2^*}{z_2 z_2^*} = \frac{z_1 z_2^*}{|z_2|^2}$$

6. Powers

6.1 Definition

$z^2 = z \times z$ (*) Note that : $(a+bi)^n \neq a^n + b^n i$

$z^3 = z^2 \times z$

$\vdots$

$z^n = z^{n-1} \times z$

$z^{-n} = 1/z^n$

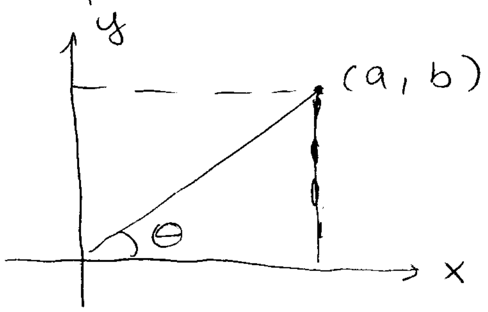
$z^0 = 1$, if $z \neq 0$

II. The trigonometric form

1. Definition

- Complex numbers can be represented as points in the xy plane.

Example: consider $z = a + ib$. The point with coordinates (a,b) represents z



This representation is known as "Argand diagram".

- Thereby, the cartesian coordinates (a,b) represent $Re[z]$ and $Im[z]$

And now about the polar coordinates?

Polar coordinates

$a = r\cos\theta$, with $r = \sqrt{a^2+b^2}$
$b = r\sin\theta$

Hence
$z = a+ib = \sqrt{a^2+b^2}\,(\cos\theta + i\sin\theta)$

But we have seen that $|z| = \sqrt{a^2+b^2}$

Then $z = |z|(\cos\theta + i\sin\theta)$ (this expression is called "the trigonometric form," whereas $z = a+ib$ is the "algebraic form").

- θ = argument of z $\quad \theta = \arctan\left(\frac{b}{a}\right)$
  r = modulus of z

2. Properties

2.1. $\cos(\alpha+\beta) + i\sin(\alpha+\beta) = \underbrace{(\cos\alpha + i\sin\alpha)}_{(*)}\underbrace{(\cos\beta + i\sin\beta)}_{(**)}$

Proof: if one multiplies $(*)$ and $(**)$ one obtains

$$\cos\alpha\cos\beta + i\sin\beta\cos\alpha + \underbrace{i^2}_{-1}\sin\alpha\sin\beta + i\sin\alpha\cos\beta =$$
$$= \underbrace{(\cos\alpha\cos\beta - \sin\alpha\sin\beta)}_{(****)} + i\underbrace{(\sin\beta\cos\alpha + \sin\alpha\cos\beta)}_{(***)}$$

From the trigonometric addition formulae,

$(***) = \sin(\alpha+\beta)$
$(****) = \cos(\alpha+\beta)$

2.2 - De Moivre's theorem: For any real number θ and any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Proof: by 2.1, $\cos(\underbrace{\theta+\dots+\theta}_{n\text{ times}=n\theta}) + i\sin(\underbrace{\theta+\dots+\theta}_{n\text{ times}=n\theta}) = \underbrace{(\cos\theta+i\sin\theta)\times\dots\times(\cos\theta+i\sin\theta)}_{n\text{ times}} = (\cos\theta+i\sin\theta)^n \Rightarrow \cos(n\theta)+i\sin(n\theta) = (\cos\theta+i\sin\theta)^n$

4. Please note: this theorem is VERY useful for computing powers of complex numbers

Example: compute $(1+i)^{10} = z^{10}$

- 1st step: write z in the trigonometric form

$$z = \sqrt{2}\,(\cos\theta + i\sin\theta) \text{ with } \theta = \text{Arc}\tan 1 = \frac{\pi}{4}$$

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

- 2nd step: apply the Moivre's theorem

$$z^{10} = (\sqrt{2})^{10}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{10} = (\sqrt{2})^{10}\left(\cos\underbrace{\frac{10\pi}{4}}_{5\pi/2} + i\sin\underbrace{\frac{10\pi}{4}}_{5\pi/2}\right)$$

But $\frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$ so that $z^{10} = \underbrace{(\sqrt{2})^{10}}_{2^5 = 32} \times \left(\underbrace{\cos\frac{\pi}{2}}_{=0} + i\underbrace{\sin\frac{\pi}{2}}_{=1}\right)$

$$\Rightarrow \boxed{z^{10} = 32i}$$

3. Complex exponentials

3.1 Euler's Formula: $e^{i\theta} = \cos\theta + i\sin\theta$ (*)

Proof: Let us write the MacLaurin's series for $\cos\theta$, $\sin\theta$ and $e^{i\theta}$

$$\cos\theta = \cos\theta\Big|_{\theta=0} + \underbrace{\frac{d(\cos\theta)}{d\theta}}_{-\sin\theta}\Big|_{\theta=0}\theta + \frac{1}{2!}\underbrace{\frac{d^2(\cos\theta)}{d\theta^2}}\Big|_{\theta=0}\theta^2 + \dots$$

$$\frac{d}{d\theta}(-\sin\theta) = -\cos\theta$$

$$\cos\theta = 1 - \frac{1}{2!}\theta^2 + \frac{\theta^4}{4!} + \dots$$

$$\sin\theta = \sin\theta\Big|_{\theta=0} + \frac{d}{d\theta}(\sin\theta)\Big|_{\theta=0}\theta + \frac{1}{2!}\underbrace{\frac{d^2}{d\theta^2}(\sin\theta)}\Big|_{\theta=0}\theta^2 + \frac{1}{3!}\frac{d^3}{d\theta^3}(\sin\theta)\Big|_{\theta=0}\theta^3 + \dots$$

$$\frac{d}{d\theta}(\cos\theta) = -\sin\theta$$

$$\sin\theta = \cos\theta\Big|_{\theta=0}\theta - \frac{1}{3!}\cos\theta\Big|_{\theta=0}\theta^3 + \dots = \theta - \frac{1}{3!}\theta^3 + \frac{\theta^5}{5!} - \frac{\theta^7}{7!}$$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$\underbrace{\cos\theta + i\sin\theta}_{(*)}$ can be written as:

$$(*) \Rightarrow 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \frac{\theta^7}{7!} + \dots\right)$$

- note that: $(i)^0 = 1$; $(i)^1 = i$, $(i)^2 = -1$, $(i)^3 = i \cdot i^2 = -i$
  $(i)^4 = (i)^2 (i)^2 = 1, \dots$

So, rearranging $(*)$, one has

$$\cos\theta + i\sin\theta = (i\theta)^0 + i\theta \underbrace{- \frac{\theta^2}{2!}}_{+(i\theta)^2} \underbrace{- \frac{i\theta^3}{3!}}_{+(i\theta)^3} + \dots = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

Hence $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$

Consequences:

(a) The trigonometric form of a complex number can be written as $z = re^{i\theta}$ $(z^* = re^{-i\theta})$

(b) The Moivre's theorem can be written as $(e^{i\theta})^n = e^{in\theta}$ for every integer n

(c) $|e^{i\theta}| = 1$ (since $\sin^2\theta + \cos^2\theta = 1$ and $|e^{i\theta}|^2 = e^{i\theta} \cdot e^{-i\theta}$)

(d) $e^{\pi i/2} = i$, $e^{\pi i} = -1$, $e^{2\pi i} = 1$

(e) $\frac{e^{+i\theta} + e^{-i\theta}}{2} = \cos\theta$; $\frac{e^{i\theta} - e^{-i\theta}}{2i} = \sin\theta$

3.2- Roots

Let us consider a complex number c. If n is a natural number and z the n-th root of c then

$$z^n = c \Rightarrow z = c^{1/n}$$

- If $c = 0 \Rightarrow z = 0$
- If $c \neq 0$, how to compute this root?

Let us write $c = re^{i\theta}$ (trigonometric form)

then $z = r^{1/n} e^{i\theta/n}$. However, one should note that

$$e^{2\pi i} = 1, \text{ so that } c = r e^{i(\theta + 2\pi)}$$

$$c = r e^{i(\theta + 4\pi)}$$

$$\vdots$$

or, in general, $c = r e^{i(\theta + 2k\pi)}$, k integer

Hence, $z_k = r^{1/n} e^{i(\theta + 2k\pi)/n}$ $k = 0, 1, 2, \ldots$

(*) Please note: $z_0, z_1, \ldots$ are all solutions of $z^n = c$.

But, for $k > n$, the roots repeat themselves

$$(e^{2\pi i} = 1 \quad \text{so} \quad z_n = e^{i\theta + 2\pi i} = e^{i\theta} = z_0$$

$$\vdots$$

$$z_{n+k} = e^{i\theta + 2\pi i + \frac{2\pi i k}{n}} = e^{i\theta + \frac{2\pi i k}{n}} = z_k$$

Consequence: $z_0, z_1, \ldots z_{n-1}$ are all different and these are the solutions of our equation

Example: Find all solutions of the equation

$$z^3 = 27$$

(7)

* Trigonometric form:

$27 = 27 . e^{0}$, or still, $27 = 3 \times e^{0} \times e^{2\pi i k}$

$z^3 = 27 \Rightarrow z = (27)^{1/3} = (3^3 \times e^{2ik\pi})^{1/3} = 3\, e^{2ik\pi/3}$

Roots: $k = 0 \Rightarrow z_0 = 3$

$k = 1 \Rightarrow z_1 = 3\, e^{2i\pi/3} = 3\left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)\right)$

$k = 2 \Rightarrow z_2 = 3\, e^{4i\pi/3} = 3\left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)$

Explicitly: $z_1 = 3 \times \left(\underbrace{\cos\left(\pi - \frac{\pi}{3}\right)}_{-\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}} + i \underbrace{\sin\left(\pi - \frac{\pi}{3}\right)}_{\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}}\right)$

$z_1 = -3\left(\frac{1}{2} - i \frac{\sqrt{3}}{2}\right)$

$z_2 = 3\left(\underbrace{\cos\left(\pi + \frac{\pi}{3}\right)}_{-\cos\frac{\pi}{3} = -\frac{1}{2}} + i \underbrace{\sin\left(\pi + \frac{\pi}{3}\right)}_{-\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}}\right)$

$\Rightarrow z_2 = -3\left(\frac{1}{2} + i \frac{\sqrt{3}}{2}\right)$

(*) Please note: apart from the real root 3, there exist other two distinct roots. Thereby, $z_2 = z_1^*$ (the roots occur in conjugate pairs)

3.2.1 - The fundamental theorem of algebra

(Gauss's PhD thesis)

Let $a_1, \dots, a_n$ be complex numbers. Then there exist complex numbers $z_1, \dots, z_n$ so that, for every complex number z,

$$z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = (z - z_1) \dots (z - z_n)$$

– (i.e., a complex polynomial of degree n has n roots)

Example: the polynomial $x^2 + 4 = 0$ has two roots. Indeed,

$$(x+2i)(x-2i) = x^2 - (2i)^2 = x^2 + 4$$

- Finding the roots: $x^2 = -4 = 4\,e^{i\pi + 2in\pi}$

$$x = 4^{1/2}(e^{i\pi/2 + in\pi})$$

$$n = 0 \Rightarrow x_1 = 2e^{i\pi/2} = 2i$$

$$n = 1 \Rightarrow x_2 = 2e^{i3\pi/2} = -2i$$

(*) Please note: if a_n are real, the roots ALWAYS occur in conjugate pairs.

4 - Remarks on calculus

Consider $f(t) = g(t) + i h(t)$

$$\int f(t)\,dt = \int g(t)\,dt + i \int h(t)\,dt$$

$$f'(t) = g'(t) + i h'(t)$$

(the differentiation/integration rules remain the same)

Please note: $e^{i\theta}$ is VERY useful for computing integrals involving Trigonometric functions

Example: $\int e^{-x} \cos x \, dx = \int e^{-x} \left(\frac{e^{ix} + e^{-ix}}{2} \right) dx =$

$$= \frac{1}{2}\int e^{(-1+i)x}dx + \frac{1}{2}\int e^{(-1-i)x}dx = \frac{1}{2}\left[\frac{e^{(-1+i)x}}{(-1+i)} + \frac{e^{(-1-i)x}}{(-1-i)}\right] + c$$ (9)

$$= \frac{e^{-x}}{2}\left[\frac{(-1-i)e^{ix}+(-1+i)e^{-ix}}{\underbrace{(-1+i)(-1-i)}_{(-1)^2[1-\underbrace{(-i)^2}_{-1}]=2}}\right] + c = \frac{e^{-x}}{2}\left[-\underbrace{\frac{(e^{ix}+e^{-ix})}{2}}_{\cos x} - \underbrace{\frac{i}{2}(e^{ix}-e^{-ix})}_{=}\right] + c$$

$$-\frac{i}{2}\cdot\frac{i}{i} = \frac{1}{2i}(e^{ix}-e^{-ix}) = \sin x$$

Hence $\int e^{-x}\cos x\,dx = \frac{e^{-x}}{2}[-\cos x + \sin x] + c$