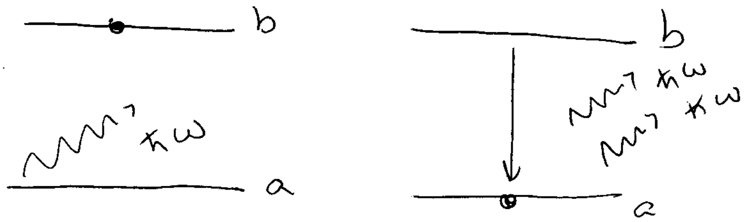
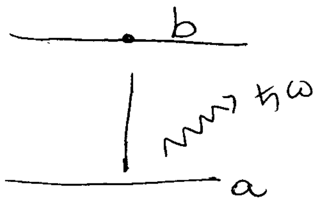
(1) Stimulated emission occurs when a photon interacts with an excited atom. This allows the atom to release a photon and decay to a lower, more stable state. The emitted photon has the same direction, energy and polarization
5 as the original photon



3 Stimulated emission may only occur in the presence of a radiation field

$A^* + \hbar\omega \rightarrow A + 2\hbar\omega$

Spontaneous emission occurs when an excited atom decays
5 to a lower state, emitting a photon. The direction and polarization of this photon are random.



$A^* \rightarrow A + \hbar\omega$

Stimulated emission may be described within a semiclassical
2 framework, whereas a rigorous description of spontaneous emission requires a quantum-electrodynamical treatment. Physically, this is due to the fact that the vacuum state
5 can mix with the stationary states of the atom. Hence, the stationary state of the atom is no longer a true eigenstate of the combined system atom + field, and decay may occur.

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2 The dipole approximation consists in neglecting the spatial dependence of an electromagnetic field. If one considers, for instance, the matrix element

$M_{ba} = \langle \Psi_b | \exp(i\vec{k}\cdot\vec{r})\, \hat{\epsilon}\, \nabla | \Psi_a \rangle$ describing a one-photon transition to a state b from a state a, this implies that

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dipole approximation

$\exp(i\vec{k}\cdot\vec{r}) \simeq 1 + i\vec{k}\cdot\vec{r}$

This implies that $\vec{A}(t)$, $\vec{E}(t)$ depend only on the time t, and that the magnetic field vanishes as $\nabla \times \vec{A} = 0$. The dipole approximation requires that $\vec{k}\cdot\vec{r} \ll 1$. Physically, this means that the wavelength of the radiation field must be much larger than the atomic distances involved. The dipole approximation is very good for optical transitions for which $k \cdot r \sim 10^{-3}$, but starts to break down for high frequency fields, such as X Ray radiation.

3

(a)

The operator $P_z y - P_y z$ can be written as the x component of the orbital angular momentum operator

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$$\hat{L}_x = \frac{\hat{L}_+ + \hat{L}_-}{2}$$

The operators $\hat{L}_+$ and $\hat{L}_-$, acting upon a ket $|n\, l\, m\rangle$, yield

2

$$L_+|n,l,m\rangle = \hbar\,[l(l+1) - m(m+1)]^{1/2}\,|n,l,m+1\rangle$$
$$L_-|n,l,m\rangle = \hbar\,[l(l+1) - m(m-1)]^{1/2}\,|n,l,m-1\rangle$$

Let us now consider the matrix element coupling the states $|n,l,m\rangle$ and $|n',l',m'\rangle$

3 $\langle n' \ell' m' | L_\pm | n \ell m \rangle = \hbar \left[\ell(\ell+1) - m(m\pm1) \right]^{1/2} \langle n' \ell' m' | n \ell m\pm1 \rangle$

Since the hydrogenic states $|n\ell m\rangle$ form an orthonormal basis,

2 $\langle n' \ell' m' | n \ell m\pm1 \rangle = \delta_{nn'} \delta_{\ell\ell'} \delta_{m' m\pm1}$

Hence, the matrix element is non-vanishing if

3
$$\Delta n = 0$$
$$\Delta \ell = 0 \qquad (*)$$
$$\Delta m = \pm 1$$

For the specific matrix element considered in this problem, i.e., $\langle 1,0,0 | P_z y - P_y z | 3,2,m \rangle$, $\Delta \ell = 2$

3
$$\Delta n = 2$$
$$\Delta m = m$$

4 Only the third condition in (*) can be fulfilled, for $m = \pm 1$
Hence, the matrix element vanishes
(no states are coupled) 20

(b) The result is expected, as the matrix element corresponds to a magnetic dipole transition. The selection rules for this type of transition are $\Delta n = 0$, $\Delta \ell = 0$, $\Delta m = 0, \pm 1$. 10
Hence, only states with the same principal and orbital quantum number are coupled.

4 (a) We are interested in the matrix elements

1 $\langle 1,0,0 | x y | 3,2,m \rangle$ (initial state 3d → final state 1s)

or $\langle 3,2,m | x y | 1,0,0 \rangle$ (initial state 1s → final state 3d)

We will be using properties of spherical harmonics.

Therefore, it is of interest to write x, y in such a basis

2 $$x = r\sqrt{\frac{2\pi}{3}}\left[Y_1^{-1} - Y_1^1\right]$$

$$y = ri\sqrt{\frac{2\pi}{3}}\left[Y_1^1 + Y_1^{-1}\right]$$

1 $$xy = i\left(\frac{2\pi}{3}\right) r^2 \left[\cancel{Y_1^{-1} Y_1^1} - (Y_1^1)^2 + (Y_1^{-1})^2 - \cancel{Y_1^{-1} Y_1^1}\right]$$

$$(Y_1^{\pm 1})^2 = \frac{3}{8\pi}(\sin\theta)^2 \exp[\pm i 2\phi]$$

3 Note, however, that $Y_2^{\pm 2} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}(\sin\theta)^2 \exp[\pm i 2\phi]$

$$\Rightarrow (Y_1^{\pm 1})^2 \propto Y_2^{\pm 2}$$

Matrix element (in general)

2 $$\langle n' \ell' m' | xy | n \ell m\rangle \propto \overbrace{\int_0^\infty r^4 R_{n'\ell'}(r) R_{n\ell}(r)\, dr}^{I_R \text{ (radial integral)}} \times$$

$$\times \underbrace{\int Y^{*m'}_{\ell'}(\theta,\varphi)\, Y_2^{\pm 2}(\theta,\varphi)\, Y_\ell^m(\theta,\varphi)\, d\Omega}_{I_\Omega \text{ (will give the selection rules)}}$$

1 For the problem addressed here, either the final or the initial state is an s state. This will simplify I_Ω.

Initial s state:

$$I\Omega = \int Y_{\ell'}^{*m'}(\theta,\varphi)\, Y_2^{\pm 2}(\theta,\varphi)\, \overbrace{Y_0^0}^{\text{const.}}(\theta,\varphi)\, d\Omega$$

$$\propto \int Y_{\ell'}^{*m'}(\theta,\varphi)\, Y_2^{\pm 2}(\theta,\varphi)\, d\Omega = \delta_{\ell',2}\, \delta_{m',\pm 2}$$
$$\Rightarrow \ell' = 2 \;;\; m' = \pm 2$$

5 By inspection, we see that ℓ increased in 2 units and m increased/decreased in 2 units

$$\Rightarrow \boxed{\begin{array}{l}\Delta \ell = 2\\ \Delta m = \pm 2\end{array}}$$

Final s state $\quad Y_2^{*\mp 2}(\theta,\varphi)$

5 $$I\Omega = \int Y_0^0(\theta,\varphi)\, \overbrace{Y_2^{\pm 2}(\theta,\varphi)}\, Y_\ell^m(\theta,\varphi)\, d\Omega$$

$$\propto \int Y_2^{*\mp 2}(\theta,\varphi)\, Y_\ell^m(\theta,\varphi)\, d\Omega = \delta_{\ell,2}\, \delta_{m,\mp 2}$$
$$\Rightarrow \ell = 2 \;,\; m = \mp 2$$

$$\Rightarrow \boxed{\begin{array}{l}\Delta \ell = -2\\ \Delta m = \pm 2\end{array}}$$

$$\Rightarrow \boxed{\begin{array}{l}\Delta \ell = \pm 2\\ \Delta m = \pm 2\end{array}} \checkmark$$

Alternative derivation (based on Legendre polynomials)

2 $$I\Omega = \underbrace{\int_0^{2\pi} e^{i(m-m'\pm 2)\phi}\, d\phi}_{I\phi}\; \underbrace{\int_{-1}^{1} \sin^2\theta\, P_{\ell'}^{m'}(\cos\theta)\, P_\ell^m(\cos\theta)\, d\cos\theta}_{I\theta}$$

2 $I\varphi \neq 0$ if $m - m' \pm 2 = 0 \Rightarrow \boxed{\Delta m = \pm 2}$

1 $$I\theta = \int_{-1}^{1} \sin^2\theta\, P_{\ell'}^{m'}(\cos\theta)\, P_\ell^m(\cos\theta)\, d\cos\theta$$

General case: can be done using

$$\sin\theta\, P_{\ell'}^{m}(\cos\theta) = \frac{1}{(2\ell+1)}\left[P_{\ell+1}^{m+1}(\cos\theta) - P_{\ell-1}^{m+1}(\cos\theta)\right] \quad \text{twice:}$$

(see lecture notes)

$$\sin^2\theta\, P_\ell^{m}(\cos\theta) = \frac{\sin\theta}{(2\ell+1)}\left[P_{\ell+1}^{m+1}(\cos\theta) - P_{\ell-1}^{m+1}(\cos\theta)\right]$$

$$= \frac{1}{2\ell+3}\left[P_{\ell+2}^{m+2}(\cos\theta) - P_\ell^{m+2}(\cos\theta)\right] +$$

$$+ \frac{1}{2\ell-1}\left[P_\ell^{m+2}(\cos\theta) - P_{\ell-2}^{m+2}(\cos\theta)\right]$$

$\Rightarrow$ this will lead to four integrals in I_θ, proportional to $\delta_{\ell+2,\ell'}$, $\delta_{\ell,\ell'}$ and $\delta_{\ell-2,\ell'}$

$\Rightarrow \Delta\ell = 0, \pm 2$. (general quadrupole selection rules).

We will consider, however, the situation in which either the initial or final state is an s state.

Initial s state:

$$\boxed{-} \qquad I_\theta = \int_{-1}^{1} \underbrace{\sin^2\theta}_{P_2^{\pm 2}(\cos\theta)}\, P_{\ell'}^{m'}(\cos\theta)\, d\cos\theta \propto \delta_{\ell',2} \Rightarrow \begin{matrix}\ell' = 2 \\ \ell = 0\end{matrix} \quad \Delta\ell = 2$$

Final s state:

$$\boxed{5} \qquad I_\theta = \int_{-1}^{1} P_2^{\pm 2}(\cos\theta)\, P_\ell^{m}(\cos\theta)\, d\cos\theta \propto \delta_{\ell,2}$$

$$\begin{matrix}\ell = 2 \\ \ell' = 0\end{matrix} \quad \Delta\ell = -2$$

$$\Rightarrow \boxed{\begin{matrix}\Delta\ell = \pm 2 \\ \Delta m = \pm 2\end{matrix}}$$

✓

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(b) Sublevels coupled by the transition

If we consider spin-orbit coupling, we will have the following sublevels

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1s : $l = 0$, $s = 1/2$ $\Rightarrow$ $j = 1/2$, $m_j = -\frac{1}{2}, +1/2$

$1S_{1/2}$ $(m_j = -1/2)$
$1S_{1/2}$ $(m_j = +1/2)$

3d : $l = 2$, $S = 1/2$

j goes from $|l-s|$ to $l+s$ $\Rightarrow$ $j = \frac{3}{2}, \frac{5}{2}$

Levels

2 $3D_{3/2} \rightarrow m_j = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$

$3D_{3/2}$ $(m_j = -3/2)$
$3D_{3/2}$ $(m_j = -1/2)$
$3D_{3/2}$ $(m_j = 1/2)$
$3D_{3/2}$ $(m_j = 3/2)$

$3D_{5/2} \Rightarrow m_j = \frac{-5}{2}, \frac{-3}{2}, \frac{-1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$

5

$3D_{5/2}$ $(m_j = -5/2)$
$3D_{5/2}$ $(m_j = -3/2)$
$3D_{5/2}$ $(m_j = -1/2)$
$3D_{5/2}$ $(m_j = 1/2)$
$3D_{5/2}$ $(m_j = 3/2)$
$3D_{5/2}$ $(m_j = 5/2)$

For the quadrupole transitions considered here, $\Delta m = \pm 2$

5 Starting from $1S_{1/2}(m_j = -1/2)$:

$1S_{1/2}(m_j = -\frac{1}{2}) \rightarrow 3D_{3/2}\ (m_j = \frac{3}{2})$
$\rightarrow 3D_{5/2}\ (m_j = \frac{3}{2})$
$\rightarrow 3D_{5/2}\ (m_j = -\frac{5}{2})$

$1S_{1/2}(m_j = 1/2) \rightarrow 3D_{3/2}\ (m_j = -\frac{3}{2})$
$\rightarrow 3D_{5/2}\ (m_j = 5/2)$
$\rightarrow 3D_{5/2}\ (m_j = -3/2)$

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