4 - Mode structure - laser resonators

4.(a) - Generalities

(*) So far: We have concentrated on the materials used in a laser ⇒ the laser frequency may be chosen by fixing a particular transition

- We have also seen that line broadening occurs for several reasons (Doppler broadening, finite lifetimes, pressure broadening, etc.)

(*) Solution : One may select frequencies further using resonators

(*) Ideal situation: cavity whose walls have an ~~infinitely~~ infinitely high conductivity
⇒ standing waves at discrete frequencies.

(*) Lasers :

- Open sides ⇒ diffraction losses
- Mirrors ⇒ transmission losses

⇒ Are stationary modes possible ?

- First (numerical) proof: A. G. Fox and T. Li, Bell Sys. Tech. J. 40, 453(1961); Proc. IEEE 51, 80 (1963)

They found modes whose intensity decreased due to losses in the cavity, but whose shape remained the same.

(*) Important issues (modes in a laser cavity)

- Modes have a finite lifetime ⇒ frequency width
  - Good cavity: mode widths ≪ atomic linewidths
  - Bad cavity: mode widths ≫ atomic linewidths

- Apart from the longitudinal modes in a cavity, there exist also transverse modes. They come from the fact that the waves propagating inside the cavity exhibit a transverse intensity variation (further distortions due to the specific geometry of the cavity, such as refractive indices of lenses, etc. are also present).

- Let us consider the field

$$\mathcal{E}(x,y,z) = \mathrm{Re}\left[\overset{n}{E}(x,y,z)\, e^{i(\omega t - kz) + i\,\phi(x,y,z)}\right]$$

We are looking for transverse patterns $\overset{n}{E}_{n,m}(x,y,z_0)$ at $z = z_0$ such that, if the field starts with that pattern, after a roundtrip in the cavity it will exhibit the same pattern at $z = z_0$

- The field components will be primarily polarized transverse to the propagation direction

⇒ TEM_{nm} (transverse electric modes)

4.(b) - Example - propagation of a Gaussian beam

- This type of propagation occurs, for instance, in confocal resonators formed by spherical mirrors
- The transverse modes of a light beam propagating along the z axis of the cavity are given by

$$E_{mn} = E_0 \underbrace{H_m\left(\frac{x\sqrt{2}}{w}\right) H_n\left(\frac{y\sqrt{2}}{w}\right)}_{\text{Hermite polynomials}} \underbrace{e^{-\frac{(x^2+y^2)}{w^2}}}_{\text{Gaussian beam}}$$

with $W \equiv$ spot size.

(*) Starting point: wave equation (gives the propagation of the field)

For our purposes, we can consider the field as a scalar

$$\Delta U = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} \qquad (*)$$

Monochromatic wave : $U(\vec{r},t) = U(\vec{r})\, e^{i\omega t}$

$$(*) \Rightarrow \Delta U + k^2 U = 0 \quad , k = \frac{\omega}{c}$$

(*) Assumptions

- The wave is propagating in the z direction

$$\Rightarrow U(x,y,z) = \underbrace{f(x,y,z)}_{\text{related to the fact that } f \text{ changes with } z} \underbrace{e^{-ikz}}_{\text{related to the propagation}}$$

- The transverse dependence determines the beam profile
- $\lim_{\substack{x\to\infty \\ y\to\infty}} f(x,y,z) = 0$
- Paraxial approximation

$$\left|\frac{\partial^2 f}{\partial z^2}\right| \ll \left|\frac{\partial^2 f}{\partial x^2}\right|, \left|\frac{\partial^2 f}{\partial y^2}\right|, k^2 f$$

$$\frac{\partial^2 U}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z} e^{-ikz} - ikf e^{-ikz}\right)$$

$$= \underbrace{\frac{\partial^2 f}{\partial z^2} e^{ikz}}_{\text{neglected}} - 2ik\frac{\partial f}{\partial z} e^{-ikz} - k^2 f e^{-ikz}$$

$$\Rightarrow \boxed{(*) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - 2ik\frac{\partial f}{\partial z} = 0}$$

(*) Ansatz : $f(x,y,z) = X(x)\,Y(y)\,G(z)\exp\left(-\frac{(x^2+y^2)}{F(z)}\right)$

$F(z), G(z)$ slowly varying

(yields $\lim_{\substack{x\to\infty \\ y\to\infty}} f(x,y,z) = 0$)

For a justification of why taking a Gaussian dependence based on the Huygen's principle see Haken, laser light dynamics, p. 49.

Inserting in Eq. (*) :

$$\frac{\partial^2 f}{\partial x^2} = Y(y)\,G(z)\,\frac{\partial}{\partial x}\left(\frac{dX}{dx} e^{-\frac{x^2+y^2}{F}} - \frac{2x}{F} X e^{-\frac{x^2+y^2}{F}}\right)$$

$$= \left(\frac{d^2X}{dx^2} - \frac{4x}{F}\frac{dX}{dx} - \frac{2}{F}X + \frac{4x^2}{F^2}X\right) Y(y)\,G(z)\, e^{-\frac{x^2+y^2}{F}}$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2} = X(x)\, G(z) \left(\frac{d^2 Y}{dy^2} - \frac{4y}{F}\frac{dY}{dy} - \frac{2}{F} Y + \frac{4y^2}{F^2} Y \right) e^{-\frac{x^2+y^2}{F}}$$

$$\frac{\partial f}{\partial z} = XY \left(\frac{dG}{dz} e^{-\frac{x^2+y^2}{F}} + G(z) \frac{x^2+y^2}{F^2(z)} \frac{dF}{dz} e^{-\frac{x^2+y^2}{F}} \right)$$

Inserting in (*) and dividing by $XYG\, e^{-\frac{x^2+y^2}{F}}$

yields

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{4x}{F}\frac{1}{X}\frac{dX}{dx} - \frac{2}{F} + \frac{4x^2}{F^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{4y}{F}\frac{1}{Y}\frac{dY}{dy} - \frac{2}{F} + \frac{4y^2}{F^2}$$

$$- 2ik \left(\frac{1}{G}\frac{dG}{dz} + \frac{(x^2+y^2)}{F^2}\frac{dF}{dz} \right) = 0$$

Regrouping,

$$\left(\frac{1}{X}\frac{d^2X}{dx^2} - \frac{4x}{FX}\frac{dX}{dx} \right) + \left(\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{4y}{FY}\frac{dY}{dy} \right) \left(-\frac{4}{F} - \frac{2ik}{G}\frac{dG}{dz} \right) +$$

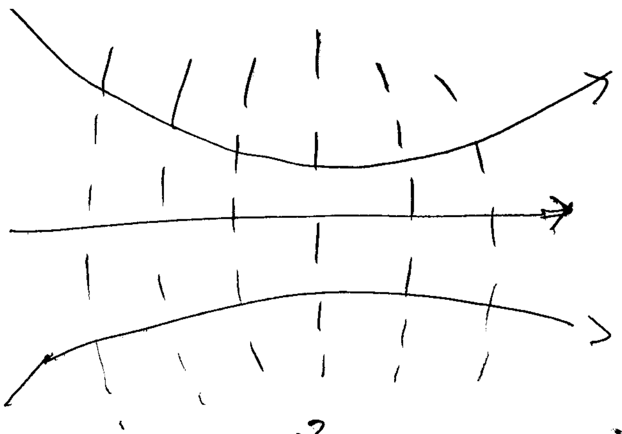
$$\left(+ 2 \frac{(x^2+y^2)}{F^2} \left(2 - ik \frac{dF}{dz} \right) \right) = 0$$

~~An~~ underlying assumptions is that F does not depend on x or y (otherwise it would influence the shape of the beam).

Hence, the last term must vanish

$$\Rightarrow \frac{dF}{dz} = \frac{2}{ik} \qquad F(z) = \frac{2}{ik}(z + C)$$

Assumption: we will have a plane wave front for $z = z_0$



This implies that the phase at z_0 cannot depend on x, y

$\Rightarrow F(z_0)$ Real $= w_0^2$

and also that

$x^2 + y^2 = w_0^2$ (otherwise we will find a gaussian dependence on x, y)

z_0

$$\frac{2}{ik}(z_0 + C) = w_0^2 \Rightarrow C = \frac{ik}{2} w_0^2 - z_0$$

$$\Rightarrow F(z) = w_0^2 + \frac{2}{ik}(z - z_0)$$

$w_0 \equiv$ minimal beam radius

Let us now consider the remaining terms

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2} - \frac{4x}{F(z)}\frac{dX}{dx}}_{\mathrm{I}} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{4y}{F(z)}\frac{dY}{dy}}_{\mathrm{II}} - \underbrace{\left[\frac{4}{F(z)} + 2ik\frac{dG/dz}{G}\right]}_{\mathrm{III}} = 0$$

The above-stated equation only holds if one can find three constants, or at most functions of F, whose sum vanishes.

If we choose $\mathrm{I} = -\frac{4m}{F}$; $\mathrm{II} = -\frac{4m}{F}$ then $\mathrm{III} = \frac{4(m+n)}{F}$

$$\Rightarrow \frac{4}{F} + \frac{2ik}{G}\frac{dG}{dz} = -\frac{4(m+n)}{F}$$

$$\frac{2ik}{G}\frac{dG}{dz} = -\frac{4(m+n+1)}{F} = -\frac{4ik(m+n+1)}{2(z+c)}$$

$$\Rightarrow \frac{dG}{G} = -(n+m+1)\frac{dz}{z+c}$$

$$\Rightarrow \ln G = -(n+m+1)\ln(z+c) + C_2$$

$$G = \frac{\hat{A}}{(z+c)^{n+m+1}}$$

Other variables:

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{4x}{F(z)}\frac{dX}{dx} = -\frac{4n}{F} \Rightarrow X(x) = H_n\left(\sqrt{\frac{2}{F}}\,x\right)$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} - \frac{4y}{F(z)}\frac{dY}{dy} = -\frac{4m}{F} \Rightarrow Y(y) = H_m\left(\sqrt{\frac{2}{F}}\,y\right)$$

$$\boxed{U_{nm}(x,y,z) = H_n\left(\sqrt{\frac{2}{F}}\,x\right) H_m\left(\sqrt{\frac{2}{F}}\,y\right)\frac{\hat{A}}{(z+c)^{n+m+1}}\, e^{-\left(\frac{x^2+y^2}{F} + ikz\right)}}$$

(*) Physical interpretation

Let us consider the exponential factor

$$\exp\left\{-\mathrm{Re}\left(\frac{1}{F}\right)(x^2+y^2) - i\left[kz + \mathrm{Im}\left(\frac{1}{F}\right)(x^2+y^2)\right]\right.$$

If we define $z_R = \frac{w_0^2 k}{2} = w_0^2 \frac{\pi}{\lambda}$ then

$$F(z) = \frac{2}{k}\left[\underbrace{z_R}_{\text{Rayleigh length}} - i(z-z_0)\right]$$

$$\frac{1}{F(z)} = \frac{k}{2} \frac{z_R + i(z-z_0)}{z_R^2 + (z-z_0)^2}$$

- Beam Radius $w(z)$ $\Rightarrow w^2(z) = \frac{1}{\mathrm{Re}\left(\frac{1}{F}\right)}$ so that

the real part of the exponential reads $\frac{x^2+y^2}{[w(z)]^2}$

$$w^2(z) = \frac{1}{\mathrm{Re}\left(\frac{1}{F}\right)} = \frac{2}{k}\left(z_R + \frac{(z-z_0)^2}{z_R}\right) = \frac{2}{k w_0^2 \pi} \lambda \cdot \frac{\pi^2 w_0^2}{\lambda^2}\left(1 + \frac{(z-z_0)^2}{z_R^2}\right)$$

$$= \cancel{\frac{2\pi}{\lambda} \cdot \frac{1}{k}}\, w_0^2 \left(1 + \frac{(z-z_0)^2}{z_R^2}\right)$$

$z - z_0 \ll z_R$ $\Rightarrow$ one can neglect the broadening in the beam. The smaller w_0 is, the shorter z_R (small aperture, large diffraction angle)

- Curvature R of the wavefront

$$\exp\left[-ik\left(z + \mathrm{Im}\left(\frac{1}{F}\right) \frac{x^2+y^2}{k}\right)\right] = \exp\left[-ik\left(z + \frac{x^2+y^2}{2R}\right)\right]$$

$$R = \frac{k}{2} \frac{1}{\mathrm{Im}\left[\frac{1}{F}\right]} = (z-z_0) + \frac{z_R^2}{(z-z_0)}$$ (curvature of the wavefront at the position z)

$z = z_0 \Rightarrow$ plane wave

(*) <u>Please note</u>

- The arguments of the Hermite polynomials are complex $\Rightarrow$ together with $\frac{1}{(z+c)^{n+m+1}}$ they give
  - the conditions for the eigenmodes in the resonator
- Transverse modes:
  - Ground mode: $n = m = 0 \Rightarrow$ gaussian intensity profile
  - Higher-order modes: the intensity profile is ~~larger~~ broader. $\Rightarrow$ larger energy loss
- Normally one selects the ground transverse mode $\Rightarrow$ higher modes lead to a messy output
- One may form a resonator by inserting at the points z_1, z_2 two reflectors with radii of curvature that match those of the propagating beam phase fronts at these points
- Alternatively, given two mirrors, one can adjust the positions z_0, and w_0 so that the mirrors coincide with the wavefronts

(*) <u>Resonance condition - longitudinal modes</u>

Let us consider the phases of a wave at $z = z_1$ and $z = z_2$

$$e^{-i\phi_1} = \frac{e^{-ikz_1}}{(z_1 - z_0 + iz_R)^{n+m+1}}$$

$$e^{-i\phi_2} = \frac{e^{-ikz_2}}{(z_2 - z_0 + iz_R)^{n+m+1}} = \frac{e^{-ikz_1}}{(z_1 - z_0 + iz_R)^{n+m+1}} \, e^{ik(z_1 - z_2)} \, \frac{(z_1 - z_0 + iz_R)^{n+m+1}}{(z_2 - z_0 + iz_R)^{n+m+1}}$$

Let us also call

$$\alpha = \frac{z_1 - z_0 + iz_R}{z_2 - z_0 + iz_R}, \quad \phi_0 = \mathrm{Arg}\, \alpha$$

$$\Rightarrow e^{-i\phi_2} = e^{-i\phi_1 + i\Delta\phi}$$

with $+\Delta\phi = k(z_1 - z_2) + (n+m+1)\phi_0$

Resonance condition: the wave has to have the same phase after a round-trip in the cavity

$$\Rightarrow \Delta\phi = n\pi$$

Depends on the cavity. Details: Yariv, Chapter 7

$$\boxed{k(z_1 - z_2) + (n+m+1)\phi_0 = n\pi}$$

If, for simplicity, we had a plane wave, the above-stated condition would reduce to

$k(z_1 - z_2) = n\pi \qquad z_1 - z_2 = l$ (cavity length)

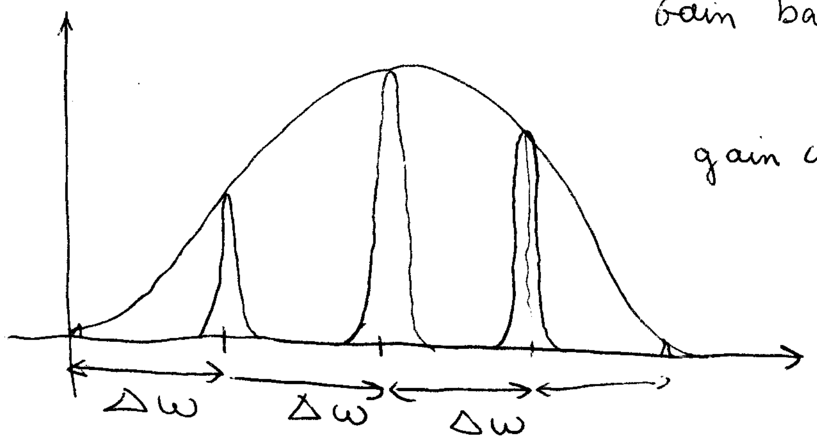
Frequency difference between two consecutive modes:

$$\omega_n \ell = n\pi c \qquad \Rightarrow \Delta\omega = \frac{\pi c}{\ell}$$
$$\omega_{n+1} \ell = (n+1)\pi c$$

Please note: these modes will also exhibit a linewidth (the cavity is an open system, there are losses, etc)

<u>Example</u>

Generic case:



Gain bandwidth = range of frequencies a laser may operate over

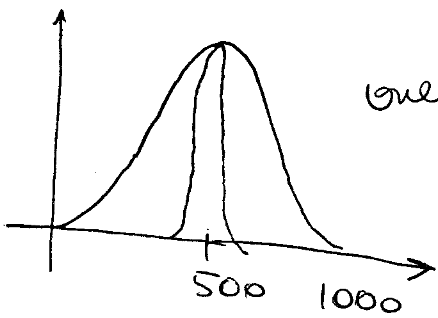
gain curve $g(\omega)$

3 modes oscillate

He - Ne laser: gain curve doppler broadened 1000 MHz

$$\Delta\omega = \frac{c\pi}{\ell} \Rightarrow \Delta\omega = 500 \text{ MHz}$$

ℓ ← 0.3 m



only one mode oscillates