Solution - Coursework 2 - 2011 - Atom - Photon Physics

1 (a) In a cavity, an electromagnetic wave bounces back and forth, so that stationary patterns build up. These stationary patterns are called "modes" (2 marks). Transverse modes are stationary patterns that occur perpendicular to the propagation of the laser field (1 mark). They are caused by the fact that the waves propagating inside a cavity exhibit a transverse intensity variation, together with the boundary conditions imposed by the geometry of the cavity (2 marks). Longitudinal modes are stationary patterns that occur in the propagation direction of the beam (1 mark). In a good or in a bad cavity, the widths of the cavity modes are much smaller, or larger, than the atomic linewidths involved, respectively (1 mark). The reason behind it is that line broadening occurs for several reasons, such as collisions, the Doppler effect, etc., and a resonator may provide a further means for frequency selection (2 marks). Apart from that, in general, one wishes to avoid many modes within the line profile, as these modes may compete (1 mark).

(b) At two points z_1, z_2 in a cavity, the radii of the mirrors should match the radii of the propagating wavefronts (1 mark). This would guarantee that, at each point in the mirrors, the surfaces of the wavefronts are normal to the directions of the energy propagation (1 mark). Consequently, the reflected beam would retrace itself and, if the phase shifts between the waves at z_1 and z_2 are a multiple of π, self-reproducing stable configurations may build up (3 marks).

Quantitatively, one may determine the curvature of a wave front at a point $z = z_1$ by using the exponential factor

$$e^{-\left[\frac{(x^2+y^2)}{F(z)} + ikz\right]}, \quad \text{where } F(z) = \frac{2}{k}\left[z_R - i(z-z_0)\right]$$

obtained in the propagation of a Gaussian beam in the paraxial approximation, where z_R is the Rayleigh length and z_0 the point along the propagation axis for which the beam has a minimum waist (at this point the beam is a plane wave) (5 marks)

The curvature R of the wave front is obtained by using

$$\exp\left[-ik\left[z + \mathrm{Im}\left[\frac{1}{F(z)}\right]\frac{(x^2+y^2)}{k}\right]\right] = \exp\left[-ik\left(z + \frac{x^2+y^2}{2R}\right)\right]$$

(2 marks)

Since $\dfrac{1}{F(z)} = \dfrac{k}{2}\,\dfrac{z_R + i(z-z_0)}{z_R^2 + (z-z_0)^2}$ (1 mark)

We find $R = \dfrac{k}{2}\,\dfrac{1}{\mathrm{Im}\left[\frac{1}{F}\right]} = (z-z_0) + \dfrac{z_R^2}{(z-z_0)}$ (1 mark)

(2) Laser light has very particular characteristics. First, it is monochromatic (1 mark), since only light from a transition between a single pair of levels is amplified (1 mark). Furthermore, when leaving the cavity, this light emerges in a narrow beam perpendicular to the plane of the mirrors (1 mark). Hence, it has directionality (1 mark). It is also highly coherent (1 mark), due to the fact that stimulated emission is the physical mechanism behind it. (1 mark). This implies that, each each transition, one photon is added to a mode of the resonator (1 mark). Hence, it is completely in phase with the other photons and has the same polarization (1 mark). Finally, laser light also exhibits high brightness, because the power output is concentrated in a narrow directional beam. (2 marks).

All these properties are in contrast to those of the light generated in a heated, incandescent filament. First, incandescent light exhibits a broad frequency distribution, as it is generated by electrons being decelerated and/or colliding with ions in the filament (2 marks). Second, it has no phase correlation and no directionality, as no extra photons are being "added" to a mode in this case (1 mark).

The properties in the laser light (especially coherence and brightness) have made it possible to reach much higher intensities than those reached by conventional light. In this intensity regime, matter may respond non-linearly to a field. (2 marks)

(3)

(a) The natural linewidth of an atomic level comes from the fact that it decays with a lifetime τ (1 mark). This finite lifetime introduces an energy width in the level, which, according to the uncertainty relation, is of the order $\Delta\omega \sim \frac{\hbar}{\tau}$ (2 marks). No, one cannot avoid this energy width, as an infinitely sharp energy level would imply an infinite lifetime according to the uncertainty relation (2 marks).

(b) A naturally broadened line exhibits a Lorentzian shape. (1 mark). It is easy to show this by considering a two-level atom interacting with an external field in the framework of first-order perturbation theory (1 mark), and assuming that, for the upper level, the population amplitude decays according to $c_b(t) = \exp[-t/(2\tau_b)]$ (1 mark) where τ_b is the natural lifetime of level b (1 mark).

Let us now consider the expression

$$\dot{c}_a(\omega;t) \propto \underbrace{M_{ab}(\omega)}_{\text{matrix element } M_{Ab} \text{ coupling the two levels (1 mark)}} \exp[i(\underset{\text{frequency of the radiation field (1 mark)}}{\omega} - \underset{\text{frequency difference between a and b (1 mark)}}{\omega_{ba}})t] \exp[-t/(2\tau_b)]$$

(1 mark for right expression) $\left(\omega_{ba} = \frac{E_b - E_a}{\hbar}\right)$

relating the amplitudes of the ground state $|\psi_a\rangle$ and of the excited state $|\psi_b\rangle$

Inserting $c_b(t) = \exp[-t/(2\tau_b)]$ we have

$$\dot{c}_a(\omega,t) \propto M_{ab}(\omega)\ \exp[i(\omega-\omega_{ba})t]\ \exp[-t/(2\tau_b)]$$

(1 mark)

Integrating,

$$c_a(\omega,t) \propto \int_0^t \exp[i(\omega-\omega_{ba})t' - t'/(2\tau_b)]\,dt' \qquad (1\ \text{mark})$$

$$= \frac{\exp[-i(\omega-\omega_{ba})t - t/(2\tau_b)] - 1}{i(\omega-\omega_{ba}) - 1/(2\tau_b)} \qquad (2\ \text{marks})$$

The probability that a one-photon transition from b to a occurs is

$$|c_a(\omega,t)|^2 \propto \left|\frac{1}{i(\omega-\omega_{ba}) - (1/(2\tau_b))}\right|^2 = \frac{1}{(\omega-\omega_{ba})^2 + 1/4\tau_b^2} \qquad (1\ \text{marks})$$

$|c_a(\omega,t)|^2$, has a maximum at $\omega = \omega_{ba}$ and decreases to a half maximum when

$$\omega = \omega_{ba} \pm 1/(2\tau_b) = (E_b - E_a \pm \Gamma_b/2)/\hbar$$

($\Gamma_b = \frac{\hbar}{\tau_b}$ is the natural width of the line). (2 marks)

Using Γ_b we obtain an intensity distribution of Lorentzian shape.

4 The key idea behind hole burning is that a strong laser can be used to burn a "hole" in the Doppler-broadened profile of a spectral line (1 mark). This happens because the light intensity in a cavity can not increase indefinitely, and for strong enough fields, stimulated transitions will reduce the population inversion density (2 marks).

The above-mentioned hole will be centered at the frequency of the applied signal and exhibit a Lorentzian shape much smaller than the width of the broadened line (2 marks). This is a consequence of the fact that the strong signal will only saturate the population difference of a group of atoms whose Doppler-shifted transition frequency

$\omega_1 = \omega_0 (1 \mp \frac{v}{c})$ (1 mark) are resonant with it (1 mark for the explanation)

ω_0 — transition frequency at rest; v — velocity

This will be a much smaller group of atoms than those present in a gas in the laser cavity (1 mark). The latter will obey a Maxwellian velocity distribution

$$dN = N_0 \exp\left[-Mv^2/(2k_bT)\right] dv$$ (1 mark)

dN — Number of atoms with velocities between v and $v+dv$; M — atomic mass; k_b — Boltzman const.; T — absolute temperature

which will lead

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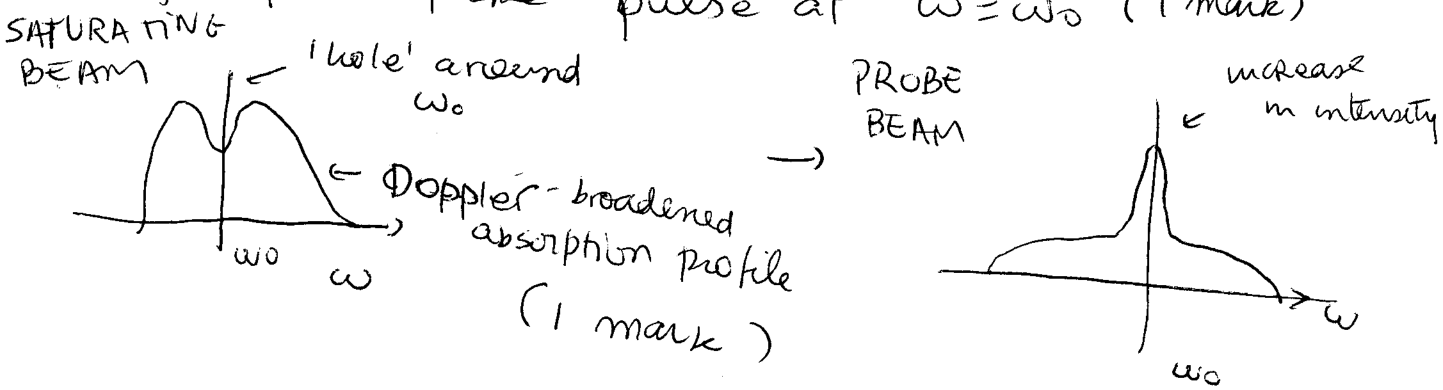
to a Gaussian intensity profile for the light emitted / absorbed by the gas, centered at $\omega = \omega_0$ (1 mark).

The idea behind saturation absorption spectroscopy is to employ a strong-saturating beam and a weak probe beam, both tuned at the rest transition frequency ω_0 (2 marks).

The strong beam will reduce the absorption of the sample around the freq. ω_0 by burning a hole in the absorption profile (1 mark). This implies that there will be an increase in the intensity of the probe pulse at $\omega = \omega_0$ (1 mark)



5 (a)

Rate equations:

$$\frac{dN_4}{dt} = W_p(N_1 - N_4) - (\gamma_{43} + \gamma_{42} + \gamma_{41}) N_4$$ (1 mark)

$$\frac{dN_3}{dt} = \gamma_{43} N_4 - (\gamma_{32} + \gamma_{31}) N_3$$ (1 mark)

$$\frac{dN_2}{dt} = \gamma_{42} N_4 + \gamma_{32} N_3 - \gamma_{21} N_2$$ (1 mark)

$$\frac{dN_1}{dt} = W_p(N_4 - N_1) + \gamma_{41} N_4 + \gamma_{31} N_3 + \gamma_{21} N_2$$ (1 mark)

The lasing transition is from level 3 to level 2 (1 mark)

(b) We are interested in the population inversion for the levels involved in the lasing transition, i.e., 2 and 3 (1 marks). Hence, we will focus on N_2, N_3 for the stationary regime in which, $\frac{dN_n}{dt} = 0$ (1 marks).

(for N_4) $\Rightarrow$ $W_p(N_1 - N_4) - (\gamma_{43} + \gamma_{42} + \gamma_{41}) N_4 = 0$

$$N_4 [W_p + (\gamma_{43} + \gamma_{42} + \gamma_{41})] = W_p N_1$$

$$\boxed{N_4 = \frac{W_p}{W_p + (\gamma_{43} + \gamma_{42} + \gamma_{41})} N_1}$$ (1 mark)

$N_3: \gamma_{43} N_4 - (\gamma_{32} + \gamma_{31}) N_3 = 0 \Rightarrow \boxed{N_3 = \frac{\gamma_{43}}{\gamma_{32}+\gamma_{31}} N_4}$

(1 mark)

$N_2: \gamma_{42} N_4 + \gamma_{32} N_3 - \gamma_{21} N_2 = 0$

$\frac{\gamma_{42}\gamma_3}{\gamma_{43}} N_3 + \gamma_{32} N_3 - \gamma_{21} N_2 = 0$ (1 mark)

$\left[\frac{\gamma_{42}\gamma_3}{\gamma_{43}} + \gamma_{32}\right] N_3 - \gamma_{21} N_2 = 0$

$N_2 = \left[\frac{\gamma_{42}(\gamma_{32}+\gamma_{31})}{\gamma_{43}\gamma_{21}} + \frac{\gamma_{32}}{\gamma_{21}}\right] N_3$ (1 mark)

Condition for population inversion: $\frac{N_2}{N_3} < 1$ (1 mark)

$\left[\frac{\gamma_{42}(\gamma_{32}+\gamma_{31})}{\gamma_{43}\gamma_{21}} + \frac{\gamma_{32}}{\gamma_{21}}\right] < 1$ (1 mark)

If we assume that the upper level will relax primarily into level 3, $\gamma_{42} << 1$ (1 mark) and the above-stated condition may be approximated by

$\frac{\gamma_{32}}{\gamma_{21}} < 1 \Rightarrow \gamma_{32} < \gamma_{21}$ (1 mark)

(c) A four-level system, which is the system discussed above, would be more efficient. (1 mark). In a three-level scheme, atoms are excited from the ground state 1 to an upper level 3, and, subsequently, decay to an intermediate level 2. (3 marks). If the decay from 3 to 2 is fast, and from 2 to 1 slow, in principle population inversion between 2 and 1 may build up (3 marks). In order, however, for this to occur, at least half of the ground-state population must be pumped into the upper state 3 (1 mark). This would require much more energy than to achieve population inversion between two intermediate levels, which is the goal in a four-level system (2 marks).