Solution - Coursework 2 - Atom-Photon Physics 2013

1. (a) **Total: 5/100.** Doppler broadening occurs due to the fact that the light emitted by a moving atom is shifted by the Doppler effect [1 mark]. Explicitly, if the atom is moving with a non-relativistic velocity v, then it emits light of angular frequency $\omega = \omega_0(1 \mp v/c)$, where ω_0 is the angular frequency of the emitter at rest, c the speed of light and v the velocit of the emitter [2 marks]. Because, in a sample with many emitters, their velocities will obey a distribution [1 mark], broadening will occur [1 mark].

   (b) **Total: 10/100.** A Doppler-broadened line exhibits a Gaussian profile [1 mark]. This is due to the fact that the atoms in a gas at absolute temperature T obey a Maxwellian distribution [1 mark]. Explicitly, the number of atoms with velocities between v and $v + dv$ is given by

$$dN = N_0 \exp[-\frac{Mv^2}{2k_BT}]dv,$$

   where M and k_B denote the atomic mass and the Boltzmann constant, respectively [2 marks]. Using

$$\mp\frac{v}{c} = \frac{\omega - \omega_0}{\omega_0}$$

   we find

$$v^2 = \left(\frac{\omega - \omega_0}{\omega_0}\right)^2 c^2$$

   [2 marks]. Hence, the intensity of the light emitted exhibits the profile

$$I(\omega) = I(\omega_0) \exp[-\frac{Mc^2}{2k_BT}\left(\frac{\omega - \omega_0}{\omega_0}\right)^2]$$

   [1 mark] centered at the rest frequency ω_0 [1 mark] and with total width at half maximum

$$\Delta\omega = \frac{2\omega_0}{c}[\frac{2k_BT}{M}\ln 2]^{1/2}$$

   [1 mark], which is much broader than the natural linewidth [1 mark].

2. **Total: 15/100.** In a laser, stimulated emission is used for amplifying light [1 mark]. This amplification, however, will only occur if population inversion is present , i.e., if the population of the upper level involved in the laseing transition is larger than that of the lower level [1 mark]. If this does not

occur, the system will behave as an absorber [1 mark]. This is easily seen if one considers a system whose ladinf transition involves an upper level 2 and a lower level 1, with energies E_2, E_1 and numbers of atoms/volume N_2, N_1, respectively. The energy density absorbed from the field is proportional to $W_{12}N_1$, where W_{12} is the absorbtion rate per atom [1 mark]. The energy density released by stimulated emission will be proportional to $W_{21}N_2$, where W_{21} = stimualte emission rate per atom [1 mark]. Since, according to the principle of the detailed balancing, $W_{21} = W_{12}$ [1 mark], only if $N_2 > N_1$ will the medium act as an amplifier [1 mark]. If the system is in thermal equilibrium, however, this will not occur as in this case

$$\frac{N_2}{N_1} = \exp\left[\frac{-(E_2 - E_1)}{k_B T}\right].$$

[1 mark].
In a three-level system, population inversion is obtained as follows. First, the medium is pumped from the ground state to a fast decaying excited state [1 mark]. This state is strongly coupled to an intermediate, slowly decaying state (e.g., a metastable state) [1 mark]. Population will then build up in this intermediate state, until inversion with regard to the ground state is achieved [1 mark]. The lasing transition will occur between the intermediate excited state and the ground state [1 mark].
In a four-level system, once more pumping takes place from the ground state to an excited state which is strongly coupled to an intermediate metastable state [1 mark]. The lasing transition, however, will be between this state and another excited state of lower energy, which is fast decaying [1 mark]. In principle, it is then possible to achieve population inversion between these two excited states with much less energy than for a three-level system [1 mark].
Please note: there is some flexibility with regard with how the marks are awarded in this question. Even ifthe line of argument stated above is incomplete, one may award some marks for the schematic representations of a 3- and 4-level atom, as long as they are properly explained and embedded in the context. Say, 1 mark for each.

3. **Total: 15/100.** The key idea behind the ammonia (NH_3) maser iis to achieve maser action/population inversion in a two-level system by population selection [1 mark]. This has been achieved bringing together the following ingredients: (i) a molecules with a permanent dipole moment D[1 mark]; (ii) a spatially inhomogeneous electric field ε[1 mark]; (iii) a cavity with an external field in resonance with the lasing transition [1 mark] and whose length has been optimized to enhance stimulated emission [1 mark]. Apart from that, the lasing transition has been chosen as a doublet in NH_3 with an energy gap in the microwave region [1 mark]. This small gap guarantees that the system may

be excited by thermal excitation [1 mark], so that the ground- and excitated state populations are approximately equal [1 mark]. The population of the ground state is removed by applying the field (ii) in order to spatially deflect the molecules in different energy levels [1 mark]. This uses the fact that, for a molecule with a permanent dipole moment, there will be forces acting upon both states, proportional to the field gradient and with opposite signs [1 mark]. The experimental set up is then adjusted in such a way that the population of molecules in the ground state will be absorbed and those in the excited state will be directed into a cavity [1 mark]. In the cavity, both ground and excited state populations $|A_1(t)|^2 = \cos^2(D\varepsilon/(2\hbar)t)$ and $|A_2(t)|^2 = \sin^2(D\varepsilon/(2\hbar)t)$ will oscillate with time [1 mark]. The ideal cavity is chosen such that stimulated emission occurs, but repopulation of fhe excited state is avoided [1 mark]. This means that its length must be $L = vT$, where v is the most propable velocity in the molecular beam [1 mark] and $T = \pi\hbar/(D\varepsilon)$ [1 mark]. (Here they do not need to show how they performed the estimate, but they should at least state what should be the choice of the cavity length).

4. **Total: 15/100**. A laser cavity may have transverse modes and longitudinal modes [1 mark]. Transverse modes are stationary patterns that occur perpendicular to the propagation direction of the wave associated with the laser field [1 mark]. They are caused by the boundary conditions imposed by the cavity, and by the fact that the laser beam exhibits a transverse intensity variation [1 mark]. Longitudinal modes are stationary patterns that occur in the propagation direction of the light beam [1 mark]. They are also caused by the boundary conditions in the cavity and are influenced by the transverse intensity profile [1 mark] . They also occur, however, for plane waves [1 mark]. In the cavity, longitudinal modes will form when the pattern obtained in a round trip is the same [1 mark]. This implies that, in this case, that the phase difference in the wave propagating along the longitudinal direction may be 2π [1 mark]. For a plane wave propagating in the z direction, this means that $\exp[ik(z + 2L)] = \exp[i(kz + 2\pi n)]$ [1 mark], which gives a cavity of length $L = n\pi/k$ [1 mark]. In a more realistic situation, one must account for the fact that the transverse intensity variation of the beam will also affect the propagation along z [1 mark]. This will introduce additional phase shifts in the propagation [1 mark]. Apart from that, this variation will lead to spherical wavefronts [1 mark]. In order to optimize propagation, one should use spherical confocal mirrors [1 mark] which should be placed in such a way that their curvature matches those of the propagating spherical waves [1 mark].

5. **Total: 10/100.** In quantum-beat spectroscopy, the system is prepared in a coherent superposition of two energetically very close states [1 mark], which are strongly coupled to a lower-enery state [1 mark]. The time dependence of the emitted radiation will be that of a decaying exponential modulated by

superposed oscillations– the so-called "quantum beats" [1 mark]. This may be understood as follows: Let us consider two energetically close levels $|1\rangle$ and $|2\rangle$, which may decay to a lower level $|0\rangle$. After some time t has elapsed, the time-dependent wavefunction of this coherent superposition reads as

$$|\psi(t)\rangle = C_1 |1\rangle \exp[-iE_1 t - t/(2\tau)] + C_2 |2\rangle \exp[-iE_2 t - t/(2\tau)],$$

where E_1 and E_2 are the bound-state energies associated with $|1\rangle$ and $|2\rangle$ and τ the lifetime of these states [2 marks]. The intensity of the decay radiation is proportional to the transition probability from this coherent superposition to the lower level $|0\rangle$ [1 mark], which, in the dipole approximation, is proportional to

$$\begin{aligned} |\langle 0|\hat{\epsilon}\cdot\vec{D}|\psi(t)\rangle|^2 &= e^{-t/\tau}[|C_1|^2|\langle 0|\hat{\epsilon}\cdot\vec{D}|1\rangle|^2 + |C_2|^2|\langle 0|\hat{\epsilon}\cdot\vec{D}|2\rangle|^2 \\ &\quad +2|C_1||C_2|\langle 0|\hat{\epsilon}\cdot\vec{D}|1\rangle\langle 0|\hat{\epsilon}\cdot\vec{D}|2\rangle\cos[\left(\frac{E_2-E_1}{\hbar}\right)t+\phi]], \end{aligned}$$

where $\vec{D}, \hat{\epsilon}$ and ϕdenote the dipole operator, the polarization vector and a phase shift that may occur [1 mark]. This equation clearly shows an exponential decay superimposed by oscillations of angular frequency$(E_2 - E_1)$ / $\hbar$, i.e., proportional the the energy difference between the two excited levels [2 marks]. Note that for an incoherent superposition this term would be absent and this measurement would not be possible [1 mark].

6. **Saturation in a two-level atom.**

   (a) Rate equations. **Total: 5/100**

$$\frac{dN_1}{dt} = -\frac{dN_2}{dt} = -[W_{12}+\gamma_{12}]N_1(t) + [W_{21}+\gamma_{21}]N_2(t)$$

[1 mark].

$$\frac{d\Delta N(t)}{dt} = \frac{d}{dt}\left[N_1(t) - N_2(t)\right] = \frac{dN_1}{dt}$$

[1 mark]. Using $N_1 + N_2 = N$ and $N_1 - N_2 = \Delta N$, we have

$$N_2 = (N - \Delta N)/2; N_1 = (N + \Delta N)/2$$

[1 mark]. This gives

$$\begin{aligned} \frac{d\Delta N(t)}{dt} &= -\left[W_{12}+\gamma_{12}\right]\left[N+\Delta N)\right] + \left[W_{21}+\gamma_{21}\right]\left[N-\Delta N)\right] \\ &= -(W_{21}+W_{12})\Delta N(t) + (\gamma_{21}-\gamma_{12})N - (\gamma_{21}+\gamma_{12})\Delta N(t) \\ &= -(W_{21}+W_{12})\Delta N(t) - (\gamma_{12}+\gamma_{21})\left[\Delta N(t) - \frac{(\gamma_{21}-\gamma_{12})}{\gamma_{21}+\gamma_{12}}N\right] \end{aligned}$$

[2 marks; full marks only if the manipulations to obtain this result are shown; if there are no manipulations, no marks]

(b) **Total: 10/100.** Using

$$\frac{1}{T_1} = \gamma_{21} + \gamma_{12},$$

$$\Delta N_0 = N_{10} - N_{20} = \frac{(\gamma_{21} - \gamma_{12})}{\gamma_{21} + \gamma_{12}} N$$

and

$$W_{12} = W_{21}(\text{principle of the detailed balancing}),$$

we find

$$\frac{d\Delta N(t)}{dt} = \underbrace{-2W_{12}\Delta N(t)}_{(i)} \underbrace{- \frac{1}{T_1}\left[\Delta N(t) - \Delta N_0\right]}_{(ii)}$$

[2 marks]. (i) is a saturation term due to the presence of the stimulated signal [1 mark]. It will drive the population difference $\Delta N(t)$ towards zero, i.e., the system will be driven towards saturation [1 mark]. This term is proportional to the strength of the applied signal via the stimulated-transition probability W_{12} [1 mark]. This means that the stronger the field is, the faster the population difference will saturate [1 mark].Term (ii) is a relaxation term [1 mark]. It makes the population difference relax towards its thermal-equilibrium value ΔN_0[1 mark] if there is no applied signal (i.e., if $W_{12} = 0$) [1 mark]. This will happen with an exponential time constant T_1 [1 mark]

(c) **Total: 15/100.** We will now solve the differential equation in (b) with the initial condition $\Delta N(0) = \Delta N_0$ [1 mark]. This equation is of the form

$$\frac{dy(t)}{dt} + f_1(t)y = f_2(t),$$

[1 mark] whose solution is

$$y(t) = \frac{\int f_2(t) \exp[\int^t f_1(t')dt']dt + c}{\exp[\int f_1(t)dt}$$

[1 mark]. Hence

$$\Delta N(t) = \exp[-\frac{(1 + 2W_{12}T_1)t}{T_1}] \left[\frac{\exp[\frac{(1+2W_{12}T_1)t}{T_1}]}{1 + 2W_{12}T_1}\Delta N_0 + C\right]$$

[1 mark]. Unsing the initial condition stated above gives

$$C = \Delta N_0 \left[\frac{2W_{12}T_1}{1 + 2W_{12}T_1}\right]$$

[1 mark]. Thus

$$\Delta N(t) = \frac{1}{1+2W_{12}T_1}\left[\Delta N_0 + 2\Delta N_0 W_{12}T_1 \exp[-\frac{(1+2W_{12}T_1)t}{T_1}]\right]$$

[1 mark].

i. Behavior for $W_{12} \to \infty$: In this case, $\lim_{W_{12}\to\infty} \Delta N(t) = 0$ [1 mark]. This implies that an in creasing intensity in the applied signal will cause a decrease in population difference until, for an infinite intensity, the population is equally distributed in the lower and upper levels [1 mark]. For that reason, population inversion can never be achieved in a two-level atom, regardless of the intensity of the applied signal [1 mark], unless the ground-state population is removed by another mechanism [1 mark].

ii. If $t \to \infty$, the exponentially decaying term in $\Delta N(t)$ will vanish [1 mark]. This gives

$$\lim_{t\to\infty} \Delta N(t) = \frac{\Delta N_0}{1+2W_{12}T_1}$$

[1 mark]. This is the stationary value for the population difference, which occurs when there is an equilibrium between the stimulated transition term, which tends to equalize the populations bt transferring atoms from the lower to the upper state [1 mark], and the relaxation term, which tends to bring ΔN back to its thermal equilibrium value ΔN_0 [1 mark]. This result can also be obtained by setting $\Delta N(t) = 0$ in the above-stated diifferential equation [1 mark]. This gives

$$\begin{aligned} 0 &= -\frac{2W_{12}T_1\Delta N_s(t)}{T_1} - \frac{\Delta N_s(t)}{T_1} + \frac{\Delta N_0}{T_1} \\ \therefore\ \Delta N_s(t) &= \frac{\Delta N_0}{1+2W_{12}T_1}. \end{aligned}$$

[1 mark]. Note that $\Delta N_s < \Delta N$ [1 mark].