IV - Coherence

⊛ key ideas

- Every electromagnetic field has fluctuations. Real light sources, for instance, have a spatial extension, undergo irregular fluctuations in intensity, phase, etc.
- The existence of such fluctuations can be inferred from experiments probing correlations between them at two or more space-time points (interference experiments)
- Optical coherence theory: studies optical correlation phenomena / the statistical description of optical fields

⇒ Precise measure of the correlation between fluctuating fields

⊛ Definition: Let us consider two sources, emitting the fields

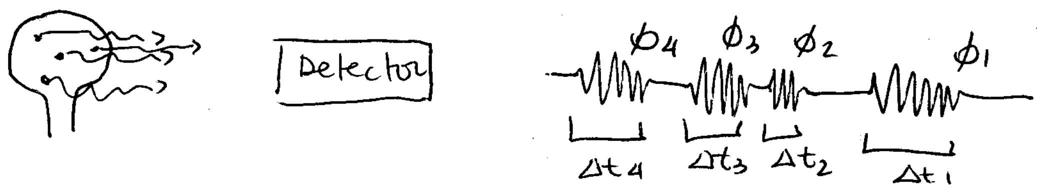
$$E_1 = E_{01} \exp[i(\vec{k}_1 \cdot \vec{r} - \omega t + \phi_1)] + c.c.$$

$$E_2 = E_{01} \exp[i(\vec{k}_2 \cdot \vec{r} - \omega t + \phi_2)] + c.c.$$

$\phi_1 - \phi_2 = \text{const}$ ⇒ the sources are mutually coherent

1 - Temporal coherence

Let us consider a quasi monochromatic source, emitting randomly phased wave trains

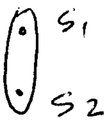


- Finite bandwidth of radiation $\Delta\nu$ exists

Fringes will be formed if $\Delta\theta\, \Delta s \leq \bar{\lambda}$

mean wavelength of the light

Explanation: Each source point S_1, S_2 gives rise to a different pattern in P, whose maxima / minima will be displaced with respect to each other

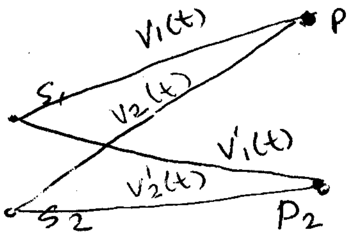


As a increases, these patterns will get more and more out of step and the fringes will disappear

Simplified model: S_1, S_2: two point sources (quasi monochromatic)

$\bar{\nu}$: mean frequency (the same)

$\Delta\nu$: effective spectral range (the same)



The sources are statistically independent

V_i $(i=1,2)$: signals reaching P_1

V_i' $(i=1,2)$: signals reaching P_2

we will asume that the differences between

$\overline{S_1 P_1}$ and $\overline{S_1 P_2}$

$\overline{S_2 P_1}$ and $\overline{S_2 P_2}$

are small compared to the coherence length

$\Rightarrow V_i'(t) = V_i(t) \quad (i=1,2) \quad (*)$

(*) Total field at P_1:

$$V(P_1, t) = V_1(t) + V_2(t)$$

(*) Total field at P_2:

$$V(P_2, t) = V_1'(t) + V_2'(t)$$

V_1 and V_2 are not correlated. Due to (*), however, their sum will.

- Observed intensity, i.e., the summed radiation, will vary slowly in frequency and amplitude

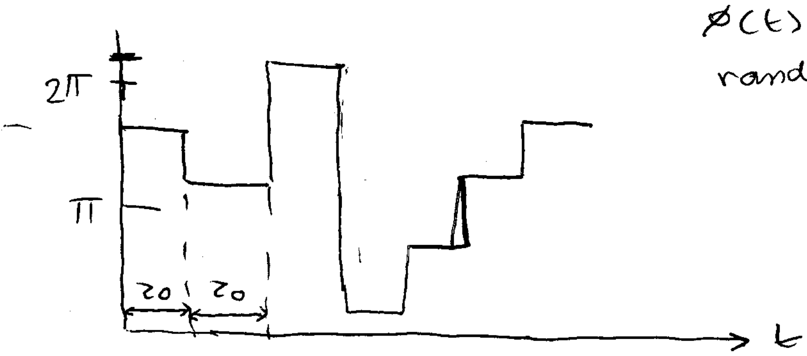
Phase-locked radiation will exist for a time τ_0,

$\tau_0 = \langle \Delta t \rangle = \frac{1}{\Delta \nu}$ is the coherence time of this light

$\equiv$ average time for which this light is coherent

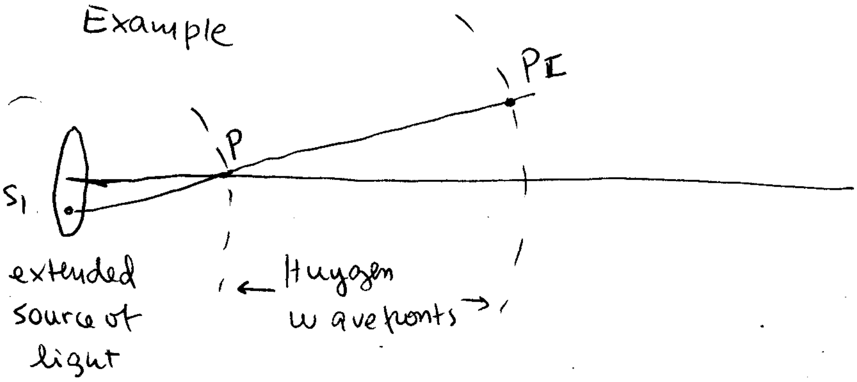
$$E(t) = Re\left[E_0 \exp\{-i\omega t + i\phi(t)\}\right]$$

$\phi(t)$ is a phase which changes randomly



Coherence length: length over which light is coherent $= l_c$

Example



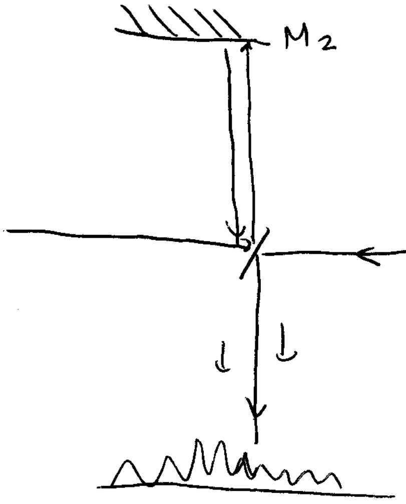
$l_c \gg PPI$: the events at P are highly correlated with those at PI

$l_c \ll PPI$: the events at P are uncorrelated with those at PI

$l_c = \tau_c\, c$ (coherence time: τ_c; coherence length: l_c)

$$\boxed{\Delta l_c = \frac{c}{\Delta \nu}}$$

Example : Michelson interferometer



Fringes occur if

$\Delta t \, \Delta \nu \lesssim 1$

$\Rightarrow \Delta t \sim \frac{1}{\Delta \nu}$ coherence time

$\Delta l \sim \frac{c}{\Delta \nu}$ coherence length

Two beams are split and reunited after a path difference $\Delta l = c \Delta t$

Physical explanation: Each frequency component will form a pattern with a different periodicity

$\Rightarrow$ With increasing time delay, their addition will lead to a less and less well-defined fringe pattern, until no pattern will be formed

Estimates (τ_c and l_c)

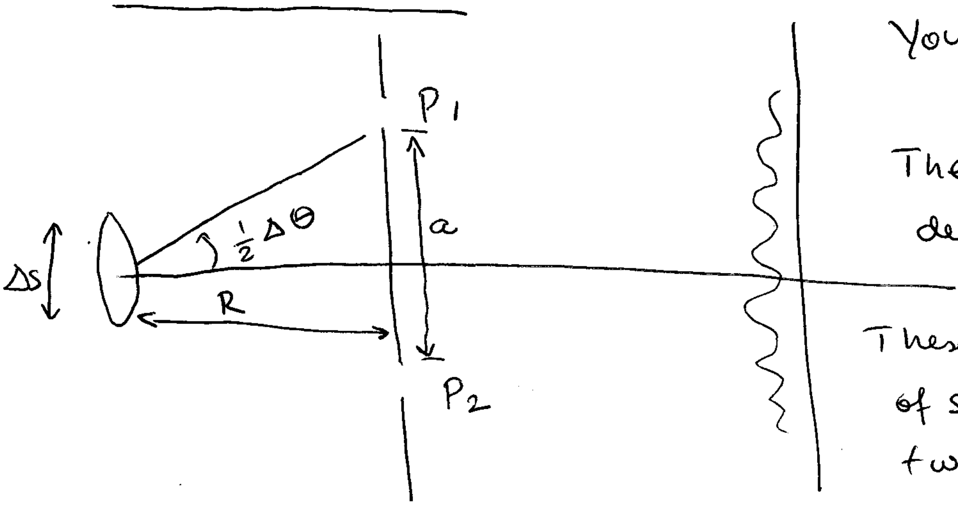
- Incandescent light: $\Delta \nu \sim 10^{8} \, s^{-1}$ $\Rightarrow \tau_c \sim 10^{-8} s$

  $\Delta l_c \sim 3 \times 10^{8} \, ms^{-1} \times 10^{-8} s = 3m$

- Laser light: $\Delta \nu \sim 10^{4} s^{-1}$ $\Rightarrow \tau_c \sim 10^{-4} s$

  $l_c \sim 30 km$

2 - Spatial coherence



Young's double slit experiment

The existence of fringes will depend on the separation a.

These patterns are a manifestation of spatial coherence between the two light beams reaching P from

P P_1, P_2.

3 - Classical theory of partial coherence

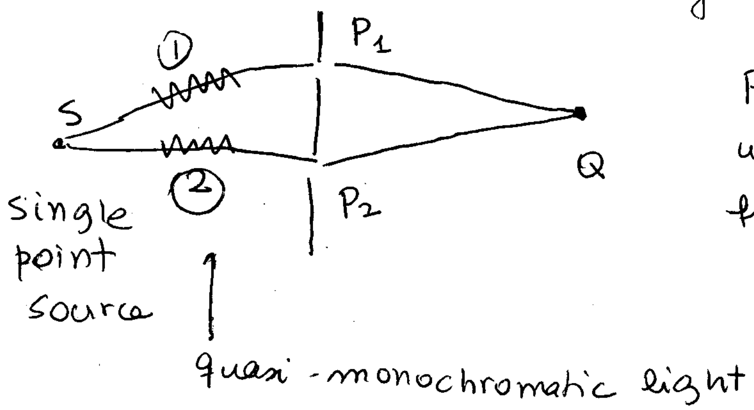
3. (a) - Temporal coherence

Coherence between two fields arriving at the same point in space through different optical paths

① Fringe visibility:

⊛ Starting point: Let us consider once more Young's double-slit experiment, but now assume that the radiation is only partly coherent.

⊛ Goal: Examine the degree of coherence in the radiation field using P_1 and P_2



Partial coherence:
we cannot write the resulting field as

$$\mathbf{E} = E_1 \sin(\omega t + \phi_1) + E_2 \sin(\omega t + \phi_2)$$

At Q we observe a time-averaged superposition of the events coming from P_1, P_2 along paths ① and ②

⊛ Observed intensity at Q:

$$I = \langle \vec{E} \cdot \vec{E}^* \rangle_t, \quad \text{where } \langle f(t) \rangle_t = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t')\, dt'$$

(temporal average)

we will consider

$$\vec{E} = k_1 \vec{E}_1 + k_2 \vec{E}_2$$

, k_1 and k_2 are propagators and depend on ① phase shifts ② amplitude reduction

(differences could be caused e.g. by lenses, absorbing materials, etc.)

$k_1 = k_2 = 1 \Rightarrow$ the media along paths (1) and (2) are identical

$$I = \langle (\vec{E}_1 + \vec{E}_2)\cdot(\vec{E}_1^* + \vec{E}_2^*) \rangle_t$$
$$= \langle |\vec{E}_1|^2 + |\vec{E}_2|^2 + 2\,\mathrm{Re}\,(\vec{E}_1 \cdot \vec{E}_2^*) \rangle_t \qquad (*)$$

* Simplifications:

- All quantities are stationary (the time average is independent of the choice of the origin of time)
- The optical fields have the same polarization, so that their vectorial nature can be ignored.

$$(*) \Rightarrow \underbrace{\langle |E_1|^2 \rangle}_{= I_1} + \underbrace{\langle |E_2|^2 \rangle}_{= I_2} + \underbrace{2\,\mathrm{Re}\langle E_1 E_2^* \rangle}_{\text{interference term (or coherence term)}} = I$$

Assumptions: the time taken by path (1) is t
the time taken by path (2) is $t+\tau$

Then the interference term is $2\,\mathrm{Re}\langle E_1(t)\,E_2^*(t+\tau)\rangle = 2\,\mathrm{Re}\,\Gamma_{12}(\tau)$

$\Gamma_{12}(\tau) \equiv$ mutual coherence function or correlation function of two fields E_1 and E_2

Similarly, the functions

$$\Gamma_{11}(\tau) = \langle E_1(t)\,E_1^*(t+\tau)\rangle$$
$$\Gamma_{22}(\tau) = \langle E_2(t)\,E_2^*(t+\tau)\rangle$$

are known as self-coherence functions or autocorrelation functions

Note that $\Gamma_{11}(0) = I_1$
$\Gamma_{22}(0) = I_2$

It is sometimes convenient to use the degree of partial coherence

$$\gamma_{12}(\tau) = \frac{\Gamma_{12}(\tau)}{\sqrt{\Gamma_{11}(0)\,\Gamma_{22}(0)}} = \frac{\Gamma_{12}(\tau)}{\sqrt{I_1 I_2}}$$

$$\therefore \quad I = I_1 + I_2 + 2\sqrt{I_1 I_2}\,\mathrm{Re}\,\gamma_{12}(\tau)$$

$\gamma_{12}(\tau)$ is a complex periodic function of τ

$\Rightarrow$ an interference pattern occurs if $|\gamma_{12}(\tau)| \neq 0$

(a) $|\gamma_{12}| = 1 \Rightarrow$ complete coherence

(b) $0 < |\gamma_{12}| < 1 \Rightarrow$ partial coherence

(c) $|\gamma_{12}| = 0 \Rightarrow$ complete incoherence

The intensity of the fringes will vary between

$$I_{max} = I_1 + I_2 + 2\sqrt{I_1 I_2}\,|\gamma_{12}|$$

$$I_{min} = I_1 + I_2 - 2\sqrt{I_1 I_2}\,|\gamma_{12}|$$

Fringe visibility $\quad V = \dfrac{I_{max} - I_{min}}{I_{max} + I_{min}} = \dfrac{2\sqrt{I_1 I_2}\,|\gamma_{12}|}{I_1 + I_2}$

$$I_1 = I_2 \Rightarrow V = \frac{2\sqrt{I_1^2}}{2I_1}\,|\gamma_{12}| = |\gamma_{12}|$$

- Complete coherence: $|\gamma_{12}| = 1 \Rightarrow$ the interference fringes have the maximum contrast of unity
- Complete incoherence: $|\gamma_{12}| = 0 \Rightarrow$ there are no fringes

(2) Self coherence of quasi monochromatic light

In $\gamma_{12}(\tau)$, $\tau \equiv$ time difference between events on paths (1) and (2)

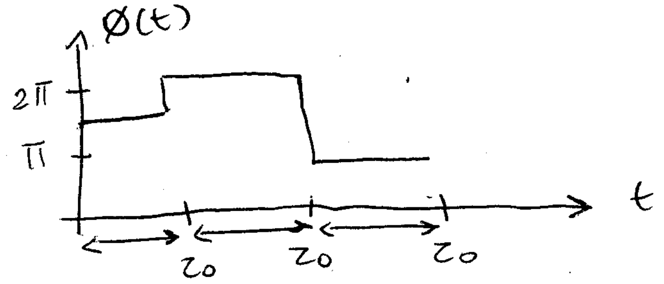
In $\gamma_{11}(\tau)$, $\tau \equiv$ time difference between two events on the same path (1).

Let us now assume that:

(1) A beam of light is divided into 2 beams to produce interference $\Rightarrow |E_1| = |E_2| = |E|$

(2) Quasi-monochromatic light: $E(t) = E_0 e^{-i\omega t} e^{i\phi(t)}$,

where ϕ is a random step function



we wish to compute the degree of self-coherence of this beam

$$\gamma_{11}(\tau) = \gamma_{22}(\tau) = \gamma(\tau)$$

$$\gamma(\tau) = \frac{\langle E(t) E^*(t+\tau) \rangle}{[\langle E(t) E^*(t) \rangle \langle E(t) E^*(t) \rangle]^{1/2}}$$

$$= \frac{\langle E(t) E^*(t+\tau) \rangle}{\langle E(t) E^*(t) \rangle}$$

$$\gamma(\tau) = \frac{\langle \cancel{E_0^2} \exp[i(\phi(t) - \omega t)] \exp[-i(\phi(t+\tau) - \omega(t+\tau))] \rangle}{\cancel{E_0^2}}$$

$$\gamma(\tau) = \exp(i\omega\tau)\,\langle \exp[i\phi(t) - \phi(t+\tau)] \rangle$$

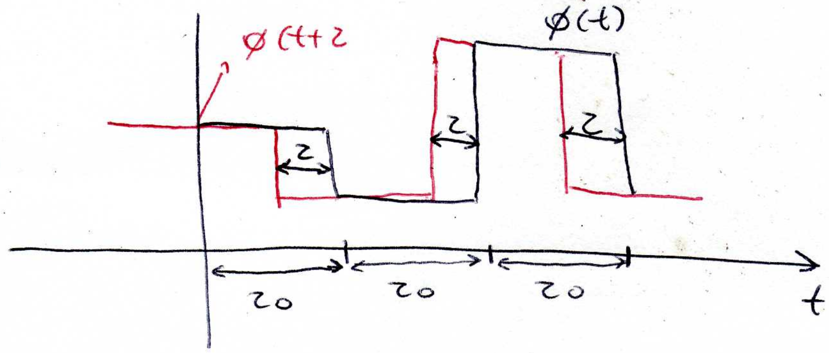
$$= \exp[i\omega\tau] \lim_{T\to\infty} \frac{1}{T}\int_0^T \exp[i(\phi(t) - \phi(t+\tau))]\,dt$$

$\tau > \tau_0$ $\Rightarrow$ The relative phases are completely random and the time average vanishes

$\gamma = 0$

$\tau < \tau_0$

Let us have a closer look at the behavior of $\phi(t) - \phi(t+\tau)$

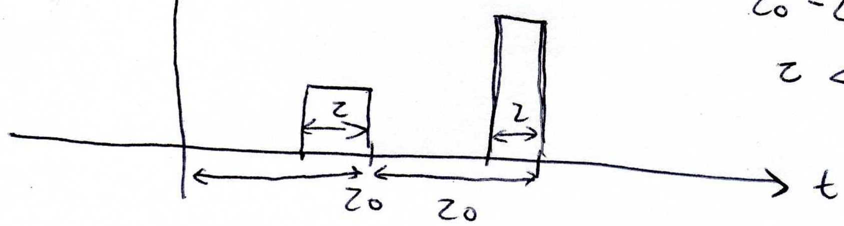


$\phi(t) - \phi(t+\tau) = \Delta\phi$

$0 < t < \tau_0 - \tau \Rightarrow \Delta\phi = 0$

$\tau_0 - \tau < t < \tau \Rightarrow \Delta\phi = \Delta$ (random value)

$\tau < t < 2\tau_0 - \tau \Rightarrow \Delta\phi = 0$



Temporal average:

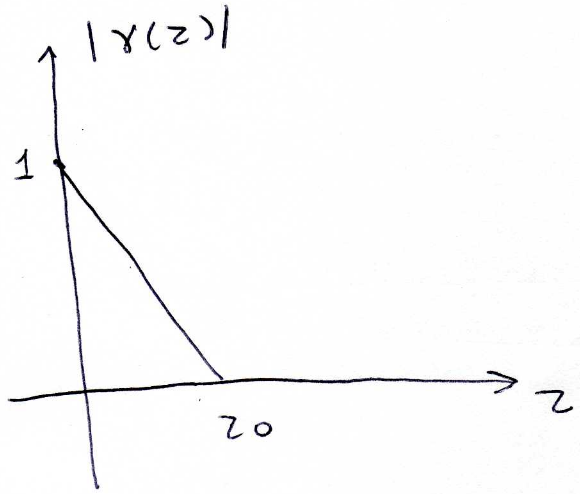
$$\frac{1}{\text{number of intervals}}\left[\frac{1}{\tau_0}\underbrace{\int_0^{\tau_0-\tau} dt}_{\tau_0-\tau} + \frac{1}{\tau_0}\underbrace{\int_{\tau_0-\tau}^{\tau_0} e^{i\Delta}\,dt}_{\tau e^{i\Delta}} + \overset{\text{other intervals}}{\cdots}\right] = \frac{\tau_0-\tau}{\tau_0} + \frac{\tau}{\tau_0}e^{i\Delta} + \dots$$

$e^{i\Delta}$ will average to zero over many intervals, as Δ

is random, while the other term is the same for all intervals

$$\gamma(z) = \begin{cases} \left(1 - \frac{z}{z_0}\right) e^{i\omega z} & z < z_0 \\ 0 & z \geq z_0 \end{cases}$$

$$|\gamma(z)| = \begin{cases} 1 - \frac{z}{z_0}, & z < z_0 \\ 0, & z \geq z_0 \end{cases}$$



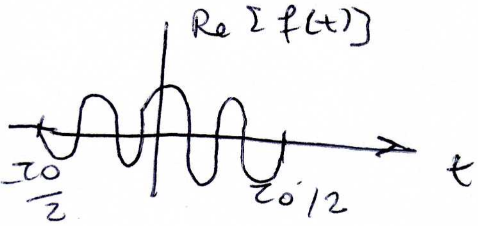
The fringe visibility drops to zero if $z > z_0$.

Coherence length:
$c z_0 = l_c$
(in this particular case, it is the length of an uninterrupted wave train)

(3) Spectral resolution of a finite wave train.

⊛ Aim: Investigate the relationship between the frequency spread and the coherence of a light source.

Let us consider a single wave train of finite duration z_0,

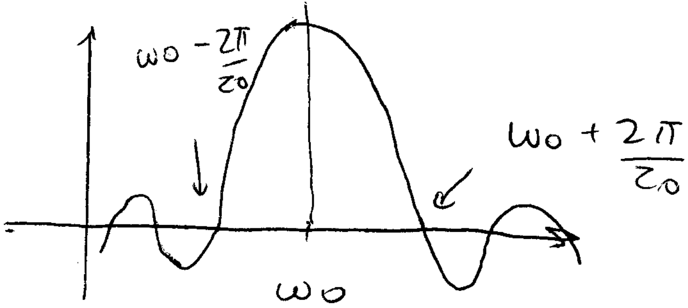


$$f(t) = \begin{cases} e^{-i\omega_0 t}, & -\frac{z_0}{2} < t < \frac{z_0}{2} \\ 0 & \text{otherwise} \end{cases}$$

The frequency spread of this wave train is given by the Fourier transform

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\tau_0/2}^{\tau_0/2} e^{i(\omega-\omega_0)t}\, dt$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin[(\omega-\omega_0)\tau_0/2]}{\omega-\omega_0}$$



Most of the energy is contained in the region between the 1st 2 minima

$$\Rightarrow \Delta\omega = \frac{2\pi}{\tau_0} \Rightarrow \Delta\nu = \frac{1}{\tau_0}$$

In a real source, $\tau_0 \to \langle \tau_0 \rangle$

$$\langle \tau_0 \rangle = \frac{1}{\Delta\nu} \qquad l_c = c\langle \tau_0 \rangle = \frac{c}{\Delta\nu}$$

$$c = \nu\lambda$$

$$\nu = \frac{c}{\lambda} \Rightarrow \Delta\nu = \frac{c\,\Delta\lambda}{\lambda^2} \Rightarrow \boxed{l_c = \frac{\lambda^2}{\Delta\lambda}}$$

Examples:

1. Discharge tubes:

$\Delta\lambda = 0.1\,nm$

$\lambda = 500\,nm \Rightarrow l_c = 2.5\,mm$

2. White light:

$\Delta\lambda = 150\,nm \Rightarrow l_c = 2\times10^3\,nm$

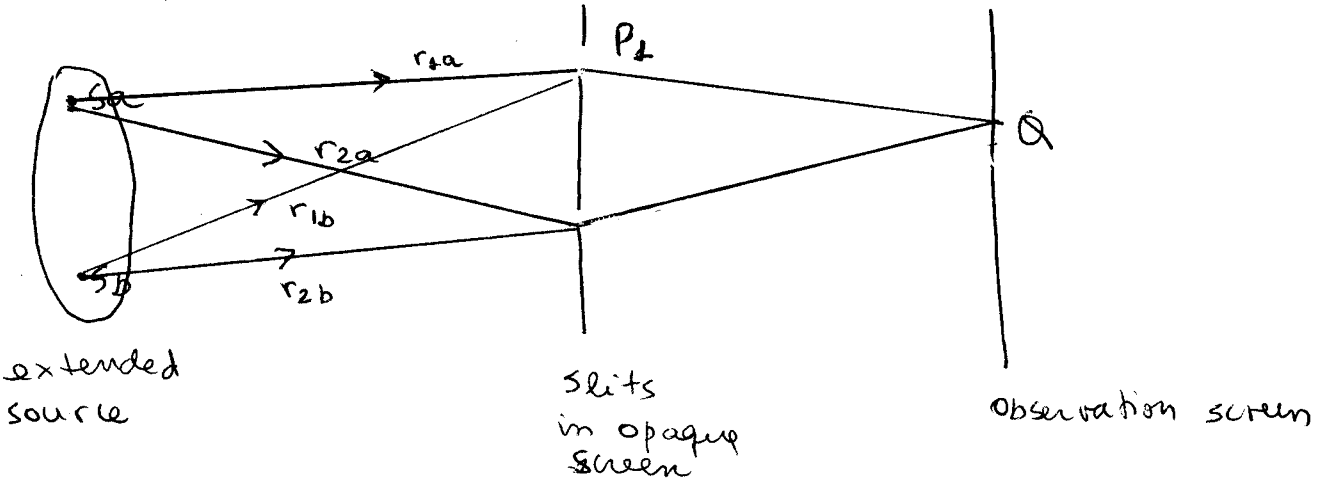
We are considering ← $\lambda = 500\,nm$ the maximum spectral sensitivity of the eye, which spans $4000\,\mathring{A} \le \lambda \le 7000\,\mathring{A}$

4λ → this is the number of fringes that can be seen, e.g., in a Michelson interferometer

3.(b) - Spatial coherence

- Coherence between two fields at different points in space
- Importance: extended sources

Let us consider the coherence of 2 points P_1, P_2 in the radiation field of an extended source



S_a, S_b are mutually incoherent and identical

The electric field at Q is $E = E_1 + E_2$

$E_1 = E_{1a} + E_{1b}$ (field at P_1)

$E_2 = E_{2a} + E_{2b}$ (field at P_2)

Degree of partial coherence for the 2 receiving points P_1, P_2:

$$\gamma_{12}(\tau) = \frac{\langle E_1(t) E_2^*(t+\tau)\rangle}{\sqrt{I_1 I_2}} = \frac{\langle [E_{1a}(t) + E_{1b}(t)][E_{2a}^*(t+\tau) + E_{2b}^*(t+\tau)]\rangle}{\sqrt{I_1 I_2}}$$

$$= \frac{\langle E_{1a}(t) E_{2a}^*(t+\tau)\rangle}{\sqrt{I_1 I_2}} + \frac{\langle E_{1b}(t) E_{2b}^*(t+\tau)\rangle}{\sqrt{I_2 I_1}} +$$

$$+ \overbrace{\frac{\langle E_{1a}(t) E_{2b}^*(t+\tau)\rangle}{\sqrt{I_1 I_2}}}^{=0} + \overbrace{\frac{\langle E_{1b}(t) E_{2a}^*(t+\tau)\rangle}{\sqrt{I_1 I_2}}}^{=0}$$

$\rightarrow$ S_a, S_b mutually incoherent

We assume now that each field E_{1a}, E_{1b}, E_{2a}, E_{2b} is of the form

$$E_{na}(t) = E_{na}\, e^{-i\omega t}\, e^{+i\phi(t)} \qquad E_{1a} = E_{1b} = E_1$$

$$E_{nb}(t) = E_{nb}\, e^{-i\omega t}\, e^{i\phi(t)} \qquad E_{2a} = E_{2b} = E_2$$

$\phi \equiv$ phase changing randomly and stepwise at a time interval τ_0

$$\frac{\langle E_{1a}(t) E^*_{2a}(t+\tau_a)\rangle}{\sqrt{I_1 I_2}} = \underbrace{\frac{\cancel{E_1 E_2}}{\cancel{\sqrt{I_1 I_2}}}}_{2E_1E_2} \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\omega t + i\omega(t+\tau_a)}$$

$$\times\, e^{i\phi(t) - i\phi(t+\tau_a)}\, dt$$

$$\underbrace{\lim_{T\to\infty} \frac{e^{i\omega\tau_a}}{T}\int e^{i\phi(t) - i\phi(t+\tau_a)}\, d\tau}$$

$$\text{''}\ e^{i\omega\tau_a}\left[1 - \frac{\tau_a}{\tau_0}\right] = \frac{\gamma(\tau_a)}{2}$$

Similarly, $\dfrac{\langle E_{1b}(t) E^*_{2b}(t+\tau_b)\rangle}{\sqrt{I_1 I_2}} = \dfrac{\gamma(\tau_b)}{2}$

$$\gamma_{12}(\tau) = \frac{1}{2}\left[\gamma(\tau_a) + \gamma(\tau_b)\right]$$

path difference time due to the path difference

$$\tau_a = \underbrace{\frac{r_{2a} - r_{1a}}{c}} + \tau$$

residual time

$$\tau_b = \frac{r_{2b} - r_{1b}}{c} + \tau$$

$$|\gamma_{12}|^2 = \frac{1}{4}\left[e^{i\omega z_a}\left(1-\frac{z_a}{z_0}\right) + e^{i\omega z_b}\left(1-\frac{z_b}{z_0}\right)\right]\left[e^{-i\omega z_a}\left(1-\frac{z_a}{z_0}\right) + e^{-i\omega z_b}\cdot\left(1-\frac{z_b}{z_0}\right)\right] = \frac{1}{4}\left[2\cos[\omega(z_a-z_b)]\left(1-\frac{z_a}{z_0}\right)\left(1-\frac{z_b}{z_0}\right) + \left(1-\frac{z_a}{z_0}\right)^2 + \left(1-\frac{z_b}{z_0}\right)^2\right]$$

Summing and subtracting $2\left(1-\frac{z_b}{z_0}\right)\left(1-\frac{z_a}{z_0}\right)$ we have

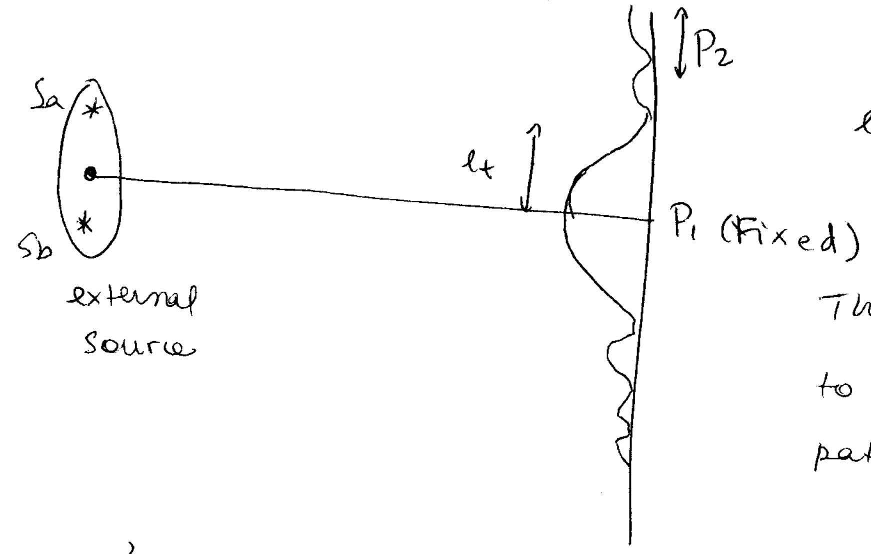
$$|\gamma_{12}|^2 = \left[\left[2+2\cos[\omega(z_a-z_b)]\right]\left(1-\frac{z_a}{z_0}\right)\left(1-\frac{z_b}{z_0}\right) \overbrace{- 2\left(1-\frac{z_a}{z_0}\right)\left(1-\frac{z_b}{z_0}\right) + \left(1-\frac{z_a}{z_0}\right)^2 + \left(1-\frac{z_b}{z_0}\right)^2}^{(*)}\right]\frac{1}{4}$$

$$(*) = \left[1-\frac{z_a}{z_0} - 1 + \frac{z_b}{z_0}\right]^2 = \left[\frac{z_b - z_a}{z_0}\right]^2 \Rightarrow \text{can be neglected if } z_b - z_a \ll z_b,\ \ll z_a$$

$$\Rightarrow \boxed{|\gamma_{12}|^2 = \frac{1}{2}\left[1+\cos[\omega(z_a-z_b)]\right]\left(1-\frac{z_a}{z_0}\right)\left(1-\frac{z_b}{z_0}\right)}$$

Consequently, the mutual coherence between the fields at P_1 and P_2 depend on

1. The self coherence time z_0 at the radiation in the source
2. The time difference $z_a - z_b$



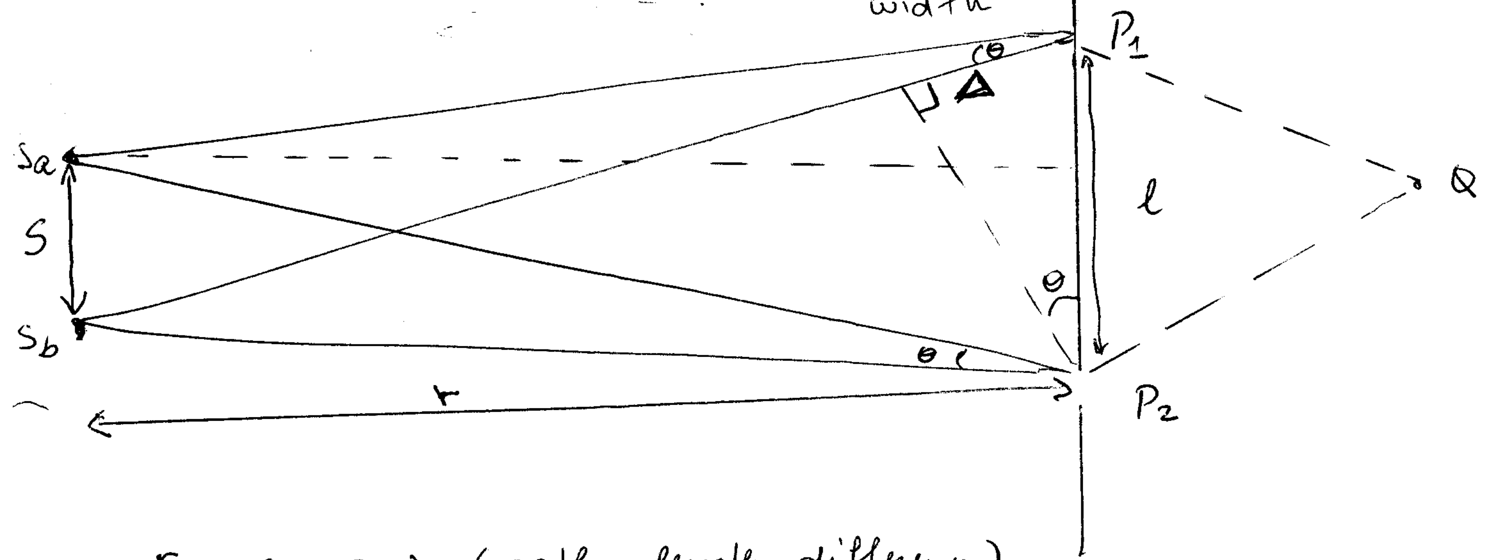
$l_t \equiv$ lateral coherence ~~length~~ width

This intensity shape is similar to a single-slit diffraction pattern.

$|\gamma_{21}|^2$ is greatest if $P_1 = P_2$

$|\gamma_{21}|^2$ drops to zero at a distance l_t such that

$\cos[\omega(\tau_b - \tau_a)] = -1 \Rightarrow \omega(\tau_b - \tau_a) = \pi \Rightarrow$ the corresponding transverse distance l_T is the transverse coherent ~~length~~ width.



$r_{b2} - r_{b1} = \Delta$ (path length difference)

$l \equiv$ distance between the slits

$S \equiv$ dist. between S_a, S_b (size of the source)

$s \simeq r\theta \quad , \quad \Delta \simeq l\theta \quad \Rightarrow \Delta = \dfrac{l s}{r}$

$\tau_a - \tau_b = \dfrac{r_{b2} - r_{b1}}{c} = \dfrac{l s}{c r} \Rightarrow \omega(\tau_b - \tau_a) = \pi$ means that

$\dfrac{\omega\, l_T\, s}{c r} = \pi$

$$\boxed{l_T = \frac{c r \pi}{\omega s}}$$

since $\omega = \frac{2\pi c}{\lambda}$, $l_t = \frac{r\lambda}{s} = \frac{\lambda}{\theta_s}$

* The Van-Cittert-Zernike Theorem

The complex degree of coherence between a fixed point P_1 and a variable point P_2 in a plane illuminated by an extended primary source is the same as the complex amplitude produced at P_2 by a spherical wave passing through an aperture of the same size and shape as the source and converging to P_1.

This is useful for computing the mutual coherence between two points of a primary source, which can be very difficult in practice

* A circular source : $l_t = \frac{1.22\lambda}{\theta}$

Example: The sun is a circular source.

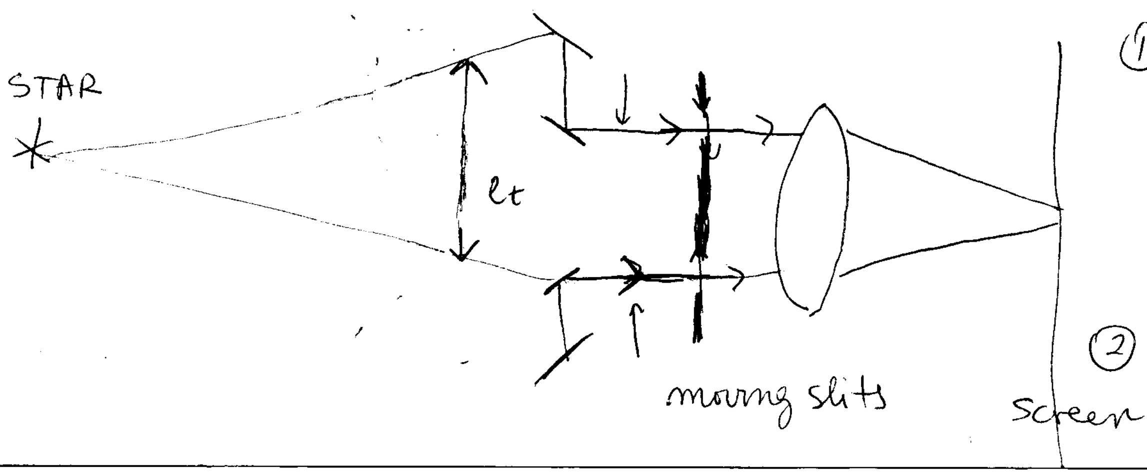
$$\theta = 9.3 \times 10^{-3} R.$$

$$l_t^{sun} = \frac{1.22 \times \overbrace{550nm}^{\text{most visible wavelength}}}{9.3 \times 10^{-3}}$$

$$l_t^{sun} = 66 \times 10^{-6} m$$

* Michelson stellar interferometer

Using the idea of a moving slit and the inherent transverse coherence of an extended circular Point Michelson was able to measure stellar distances and diameters



① l_t is determined by varying the distance between the slits (the fringes disappear

② $\theta_s = \frac{1.22\lambda}{l_t}$

(3) $\theta_s = \dfrac{\text{diameter}}{\text{distance}}$

Example : Betelgueuse

$\lambda = 570\,nm$

$\theta_s \sim 10^{-7}\,R$ $\therefore$ $l_t \sim 3\,m$