Coursework 1 - Atom-Photon Physics - Solution

1 (a)

Given: Initial state $|\psi_i\rangle = |n_1, l_1, m_1\rangle$
Final state $|\psi_f\rangle = |n_2, l_2, m_2\rangle$
coupled by an electric dipole transition
(circularly polarized light)

Prove that $\Delta l = l_2 - l_1 = \pm 1$
$\Delta m = m_2 - m_1 = \pm 1$

Electric dipole transition: the matrix element coupling $|\psi_i\rangle$ and $|\psi_f\rangle$ is

1 $$M_{fi}^{(D)} = -\frac{m}{\hbar} \omega_{fi} \hat{\epsilon} \cdot \langle \psi_f | \vec{r} | \psi_i \rangle$$

polarization vector

$$\omega_{fi} = \frac{E_f - E_i}{\hbar}$$

1 Circularly polarized light: $\hat{\epsilon}_1 = \left[\frac{\hat{\epsilon}_x + i\hat{\epsilon}_y}{\sqrt{2}}\right]$

OR

$$\hat{\epsilon}_{-1} = \left[\frac{\hat{\epsilon}_x - i\hat{\epsilon}_y}{\sqrt{2}}\right]$$

Taking

$$\vec{r} = x\hat{e}_x + y\hat{\epsilon}_x + z\hat{e}_z$$

1 $$= r\cos\theta\,\hat{\epsilon}_z + r\cos\varphi\sin\theta\,\hat{\epsilon}_y + r\sin\theta\sin\varphi\,\hat{\epsilon}_x$$

(spherical polar coordinates)

and using the fact that

$$\sin\theta\cos\phi = \sqrt{\frac{2\pi}{3}}\left[Y_1^{-1}(\theta,\phi) - Y_1^{1}(\theta,\phi)\right]$$

$\boxed{1}$ $$\sin\theta\sin\phi = i\sqrt{\frac{2\pi}{3}}\left[Y_1^{1}(\theta,\phi) + Y_1^{-1}(\theta,\phi)\right]$$

$$\cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_1^{0}(\theta,\phi)$$

We find that

$\boxed{1}$ $$\vec{r}\cdot\hat{\epsilon}_{\pm 1} = r\left(\frac{4\pi}{3}\right)^{1/2} Y_1^{\pm 1}(\theta,\phi)$$

The matrix element $\langle \psi_f | \vec{r}\cdot\hat{\epsilon}_{\pm 1} | \psi_i\rangle$

can then be written as

$$I = \left(\frac{4\pi}{3}\right)^{1/2}\int d^3r \,\langle \psi_f | \vec{r}\rangle\, r\, Y_1^{\pm 1}(\theta,\phi)\langle \vec{r}|\psi_i\rangle$$

$\boxed{2}$ with $\langle \vec{r}|\psi_i\rangle = R_{n_1 \ell_1}(r)\, Y_{\ell_1}^{m_1}(\theta,\phi)$

$$\langle \vec{r}|\psi_f\rangle = \underbrace{R_{n_2\ell_2}(r)}_{\text{Radial part of the w.f.}}\,\underbrace{Y_{\ell_2}^{m_2}(\theta,\phi)}_{\text{angular part of the w.f.}}$$

This gives

$$I = \left(\frac{4\pi}{3}\right)^{1/2} \cdot I_r \cdot I_\Omega$$

$\boxed{3}$ with $$I_r = \int_0^\infty r^3 R_{n_1\ell_1}(r)\, R_{n_2\ell_2}(r)\, dr \neq 0$$

May vanish and will give the selection rules $\rightarrow$ $$I_\Omega = \int d\Omega\; Y_{\ell_2}^{m_2 *}(\theta,\phi)\, Y_1^{\pm 1}(\theta,\phi)\, Y_{\ell_1}^{m_1}(\theta,\phi)$$

Above:
(1 mark for correct integrals and
1 mark each for each correct statement)

We will now use $Y_\ell^m(\theta,\varphi) = e^{im\varphi} P_\ell^m(\cos\theta)$

to compute the selection rules for ℓ, m.

[1] $$I\Omega = \underbrace{\int_0^{2\pi} d\varphi\, e^{-im_2\phi}\, e^{\pm i\phi}\, e^{im_1\phi}}_{I\phi} \times \underbrace{\int_{-1}^{1} d\cos\theta\, P_{\ell_2}^{m_2}(\cos\theta) \times P_1^{\pm 1}(\cos\theta)\, P_{\ell_1}^{m_1}(\cos\theta)}_{I\theta}$$

Integral over ϕ:

[1] $$\int_0^{2\pi} d\phi\, e^{+i[m_1 - m_2 \pm 1]\phi} \neq 0 \text{ only if}$$

$$m_1 - m_2 \pm 1 = 0$$

[1] $$\Rightarrow \boxed{\Delta m = m_2 - m_1 = \pm 1}$$

Integral over θ:

$$I_\theta = \int_{-1}^{1} d\cos\theta\, \sin\theta\, P_{\ell_2}^{m_1 \pm 1}(\cos\theta)\, P_{\ell_1}^{m_1}(\cos\theta)$$

We will use the fact that

[1] $$(2\ell+1)\sin\theta\, P_\ell^{m-1}(\cos\theta) = P_{\ell+1}^{m}(\cos\theta) - P_{\ell-1}^{m}(\cos\theta)$$

For $m_2 = m_1 + 1$, we have

$$\sin\theta\, P_{\ell_1}^{m_1}(\cos\theta) = \frac{1}{2\ell_1+1}\left[P_{\ell_1+1}^{m_1+1}(\cos\theta) - P_{\ell_1-1}^{m_1+1}(\cos\theta)\right]$$

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$$\Rightarrow I_\theta = \frac{1}{2\ell_1+1}\left[\underbrace{\int_{-1}^{1} d\cos\theta\, P_{\ell_2}^{m_1+1}(\cos\theta) P_{\ell_1+1}^{m_1+1}(\cos\theta)}_{I_{\theta_1}} + \right.$$

$$\left. - \underbrace{\int_{-1}^{1} d\cos\theta\, P_{\ell_2}^{m_1+1}(\cos\theta) P_{\ell_1-1}^{m_1+1}(\cos\theta)}_{I_{\theta_2}}\right]$$

Using the orthogonality relation of the associated Legendre polynomials we find that

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$$I_{\theta_1} \propto \delta_{\ell_2, \ell_1+1} \Rightarrow \Delta \ell = +1$$

$$I_{\theta_2} \propto \delta_{\ell_2, \ell_1-1} \Rightarrow \Delta \ell = -1$$

$$\boxed{\Delta \ell = \pm 1}$$

Similarly, for $m_2 = m_1 - 1$ we will use

2

$$\sin\theta\, P_{\ell_2}^{m_2} = \frac{1}{2\ell_2+1}\left[P_{\ell_2+1}^{m_2+1}(\cos\theta) - P_{\ell_2-1}^{m_2+1}(\cos\theta)\right]$$

and the same argument as above holds

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(b)

i) This transition is forbidden.

Justification: an electric dipole transition with linearly polarized light would couple states such that their orbital and magnetic quantum numbers fulfil the selection rules

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$$\Delta l = \pm 1$$
$$\Delta m = 0$$

(1 mark for the right answer, 4 marks for the justification)

For the above-stated transition, $\Delta l = +1$, but $\Delta m = +1$. Hence, the second selection rule is not satisfied and the states are not coupled.

ii) This transition is forbidden.

Justification: For a magnetic dipole transition, the principal, magnetic and orbital quantum numbers must satisfy

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$$\Delta n = 0$$
$$\Delta m = \pm 1, 0$$
$$\Delta l = 0$$

For the above-stated transition, however,

$$\Delta n = 0, \quad \Delta m = +2 \text{ and } \Delta l = +2.$$

Hence, only the condition upon n is fulfilled and the states are not coupled.

$1S_{1/2}\,(m=-1/2) \to 3D_{5/2}\,(m=-3/2)$

2 $1S_{1/2}\,(m=-1/2) \to 3D_{5/2}\,(m=+1/2)$

$1S_{1/2}\,(m=+1/2) \to 3D_{5/2}\,(m=-1/2)$ are coupled

$1S_{1/2}\,(m=+1/2) \to 3D_{5/2}\,(m=+3/2)$

1 • Component $z^2 \propto r^2 (Y_1^0)^2 \; r^2 \cos^2\theta$ $\qquad I\phi = \int_0^{2\pi} e^{im\phi - im'\phi}\, d\phi$

No dependence in m

1 states with $\Delta m = 0$ are coupled

1 $1S_{1/2}\,(m=-1/2) \to 3D_{5/2}\,(m=-1/2)$ are coupled

$1S_{1/2}\,(m=+1/2) \to 3D_{5/2}\,(m=1/2)$

• Components xy $\quad \underbrace{[Y_1^{-1}(\theta,\phi) - Y_1^1(\theta,\varphi)][Y_1^1(\theta,\varphi) + Y_1^{-1}(\theta,\varphi)]}$

2 $$xy \propto r^2\left[\underbrace{(Y_1^{-1}(\theta,\varphi))^2}_{\propto e^{-i2\varphi} \times [P_1^{-1}(\cos\theta)]^2} - \underbrace{(Y_1^1(\theta,\varphi))^2}_{e^{i2\varphi}[P_1^1(\cos\theta)]^2}\right]$$

1 Couples states with $\Delta m = \pm 2$

$1S_{1/2}\,(m=-1/2) \to 3D_{5/2}\,(m=-5/2)$

2 $1S_{1/2}\,(m=-1/2) \to 3D_{5/2}\,(m=+3/2)$ are coupled

$1S_{1/2}\,(m=+1/2) \to 3D_{5/2}\,(m=-3/2)$

$1S_{1/2}\,(m=+1/2) \to 3D_{5/2}\,(m=+5/2)$

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(ii) $1S_{1/2} \to 3D_{5/2}$, electric quadrupole transition

QUESTION: which m sublevels are coupled by the components xz, z^2, xy of the quadrupole operator?

1 For the above-stated transition, $\Delta \ell = 2$ (allowed by the electric quadrupole selection rules)

• m sublevels:

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$1S_{1/2} \to m = -\frac{1}{2}, +\frac{1}{2}$ (2 sublevels)

$3D_{5/2} \to m = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}, +\frac{5}{2}$
(6 sublevels)

Easiest way to check: write the operators in spherical harmonics and inspect the integrals in ϕ
$Y_\ell^m = e^{im\phi} P_\ell^m(\cos\theta)$

• Component xz

1
$$xz = r^2 \frac{\pi}{3} [\underbrace{Y_1^0 Y_1^{-1}}_{e^{-i \cdot 1 \phi} P_1^0(\cos\theta) P_1^{-1}(\cos\theta)} - \underbrace{Y_1^0 Y_1^1}_{e^{i \cdot 1 \phi} P_1^0(\cos\theta) P_1^{-1}(\cos\theta)}]$$

1
$$\langle n'\ell'm'|xz|n\ell m\rangle = \text{radial integral} \times \text{integral in } \theta \times \left[\int_0^{2\pi} e^{i\phi - im'\phi + im\phi} d\phi - \int_0^{2\pi} e^{-i\phi - im'\phi + im\phi} d\phi \right]$$

1
$\Rightarrow \langle n'\ell'm'|xz|n\ell m\rangle \neq 0$ for $\Delta m = \pm 1$

2 (a) x^2 operator in terms of spherical harmonics

$$x^2 \propto r^2 \left[Y_1^1(\theta,\varphi) - Y_1^{-1}(\theta,\varphi) \right]^2 = r^2 \left[(Y_1^1(\theta,\varphi)]^2 + \right.$$

1

$$\left. [Y_1^{-1}(\theta,\varphi)]^2 - 2 Y_1^1(\theta,\varphi) Y_1^{-1}(\theta,\varphi) \right]$$

Note that

$$\left[Y_1^{\pm 1}(\theta,\varphi) \right]^2 = e^{\pm i 2\varphi} \frac{1}{4}\left(\frac{3}{2\pi}\right) \sin^2\theta \propto Y_2^{\pm 2}(\theta,\varphi)$$

(Hint)

1 $$Y_1^1(\theta,\varphi) Y_1^{-1}(\theta,\varphi) = -\frac{1}{4}\left(\frac{3}{2}\right) \sin^2\theta$$

We would like to compute

$\langle n\ell m | x^2 | n00 \rangle$

1

and the selection rules will come from the angular integral

This matrix element will be proportional to

$$\text{radial integral } (\neq 0) \times \left[C_1 \int Y_2^{+2} Y_\ell^{*m}(\theta,\varphi) Y_0^0(\theta,\varphi) d\Omega \right.$$

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$$+ C_2 \int Y_2^{-2}(\theta,\varphi) Y_\ell^{*m}(\theta,\varphi) Y_0^0(\theta,\varphi)\, d\Omega \; +$$

$$\left. + C_3 \int \sin^2\theta \, Y_\ell^m(\theta,\varphi) Y_0^0(\theta,\varphi) d\Omega \right]$$

$\boxed{1}$ $$I_1 = \underbrace{\frac{1}{\sqrt{4\pi}}}_{Y_0^0} \int Y_2^{+2}(\theta,\varphi)\, Y_\ell^{*m}(\theta,\varphi)\, d\Omega \propto \underbrace{\delta_{m,+2}\,\delta_{\ell,+2}}_{\substack{\ell=2\\ m=2}}$$

$\boxed{1}$ $$I_2 = \frac{1}{\sqrt{4\pi}} \int Y_2^{-2}(\theta,\varphi)\, Y_\ell^{*m}(\theta,\varphi)\, d\Omega \propto \delta_{m,-2}\,\delta_{\ell,+2}$$

We note that the final state has $\ell=2$ and $m=\pm 2$. Since we are considering an initial s state, this implies that $\Delta\ell=+2$

$\boxed{2}$ $\Delta m = \pm 2$

Note that $\Delta\ell=-2$ would only be obtained in this case if the s state were the final state.

Computation of integral I_3

There are 2 ways of doing this. Both are provided below

Possibility 1 :

$$I_3 = \int \sin^2\theta\, Y_\ell^{*m}(\theta,\varphi)\, Y_0^0(\theta,\varphi)\, d\Omega$$

We can compute this integral for a general initial state and then take the particular case of an s state.

In this case the initial state will have

$\boxed{2}$ the angular dependence $Y_{\ell'}^{m'}(\theta,\varphi)$

we will use the fact that

$$Y_\ell^m(\theta,\varphi) = e^{im\varphi} P_\ell^m(\cos\theta)$$

1 $$I_3 = \underbrace{\int_0^{2\pi} e^{i(m'-m)\phi}\, d\phi}_{I_\varphi} \underbrace{\int_{-1}^{1} \sin^2\theta\, P_\ell^m(\cos\theta)\, P_{\ell'}^{m'}(\cos\theta)\, d\cos\theta}_{I_\theta}$$

1 Nonvanishing if
$m = m'$
$\Rightarrow \Delta m = 0$

1 Using $\sin\theta\, P_\ell^m(\cos\theta) = \frac{1}{2\ell+1}\left[P_{\ell+1}^m(\cos\theta) - P_{\ell-1}^m(\cos\theta)\right]$

the integrand in I_θ can be written as

1 $$\frac{1}{(2\ell+1)}\frac{1}{(2\ell'+1)}\left[P_{\ell+1}^m(\cos\theta) - P_{\ell-1}^m(\cos\theta)\right]\left[P_{\ell'+1}^m(\cos\theta) + - P_{\ell'-1}^m(\cos\theta)\right]$$

This will lead to the integrals

2 $$\int_{-1}^{1} P_{\ell'\pm1}^m(\cos\theta)\, P_{\ell\pm1}^m(\cos\theta)\, d\cos\theta \propto \underbrace{\delta_{\ell,\ell'}}_{\Delta\ell = 0}$$

$$\int_{-1}^{1} P_{\ell'\pm1}^m(\cos\theta)\, P_{\ell\mp1}^m(\cos\theta)\, d\cos\theta \propto \underbrace{\delta_{\ell'\pm1,\ell\mp1}}_{\Delta\ell = \pm2}$$

2 s state initial: $\ell' = 0$ so that $\Delta\ell = -2$ cannot occur
$\Delta\ell = 0$ would only hold in the very particular case that the final state is also an s state $\left[\int_{-1}^{1} \sin^2\theta\, d\cos\theta \neq 0\right]$

Sum: 20

Possibility 2:
Please note that in this case the students should justify why they made this

(b) M sublevels considering spin-orbit coupling

4d: $l = 2,\ s = 1/2 \Rightarrow j = \frac{3}{2}, 5/2$

2 $4d\,3/2\ (m = -3/2, -1/2, +1/2, +3/2)$

$4d\,5/2\ (m = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}, +5/2)$

1 2s: $l = 0,\ s = 1/2 \Rightarrow j = 1/2$

$2s\,1/2\ (m = -1/2, +1/2)$

Diagrams: levels coupled by x^2

$\Delta m = 0, \pm 2 \quad \Delta l = \pm 2$

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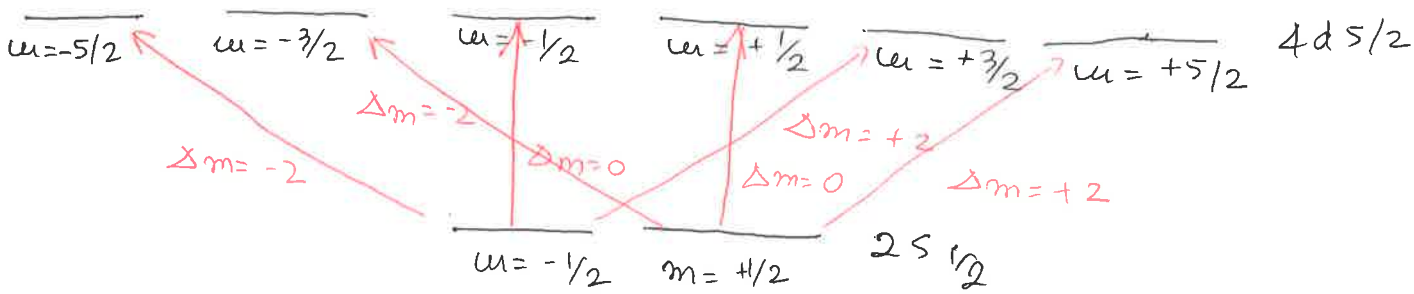

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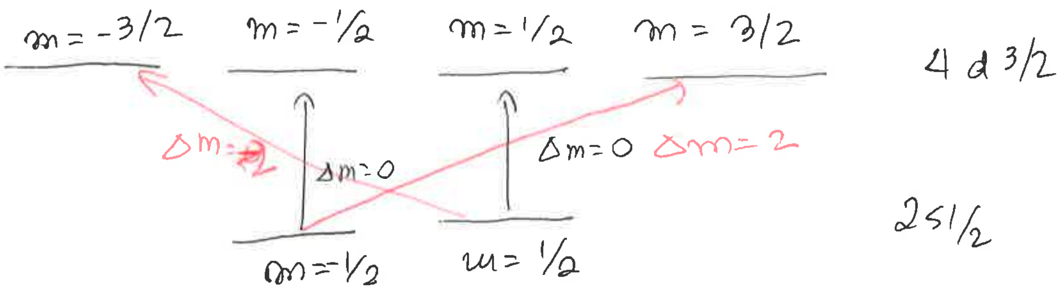

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Please note: The hint suggested that j is a good quantum number, but if the students used l instead in a correct way a few marks could be awarded.

(3) Specific properties of laser light are monochromaticity, directionality, brightness and coherence [1]. Monochromaticity occurs because light from a transition between a single pair of levels is amplified [1]. Furthermore, the laser beam is highly directional as it emerges perpendicular to the plane of the mirrors, and due to the physical mechanism involved [2]. In fact, in a laser, light is amplified by stimulated emission [1]. This means that, at each transition, one photon is being added to a pre-existing mode of the resonator [1]. Consequently, this photon will have the same properties as the other photons in that mode [1]. ~~This means~~ It will be completely in phase with them, leading to coherent light [2], and have the same direction, polarization [1].

Finally, brightness comes from the fact that the power output is highly concentrated in a narrow directional beam [1]. This property was very important for the experimental observation of non-linear phenomena, as their probability of occurrency scales with I^n (I = driving-field intensity, n = order of the process in question [2]. Hence, this occurrency is extremely low for conventional light sources [1].

In contrast, in an incandescent light bulb, light is produced by collisions in a heated filament, i.e. spontaneous emission [2]. This implies a broad spectral range, and that there will be no control over the phase, direction and polarization of the emitted photon [2]. Thus, this light will be incoherent and not directional [1]. This lack of coherence and directionality is detrimental to concentrating a large amount of energy in a small spatial region in order to obtain high brightness [1].

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(*) Please note: there is some flexibility as how the marks could be awarded as long as all the ideas stated above are presented and the student is able to bring a coherent argument.