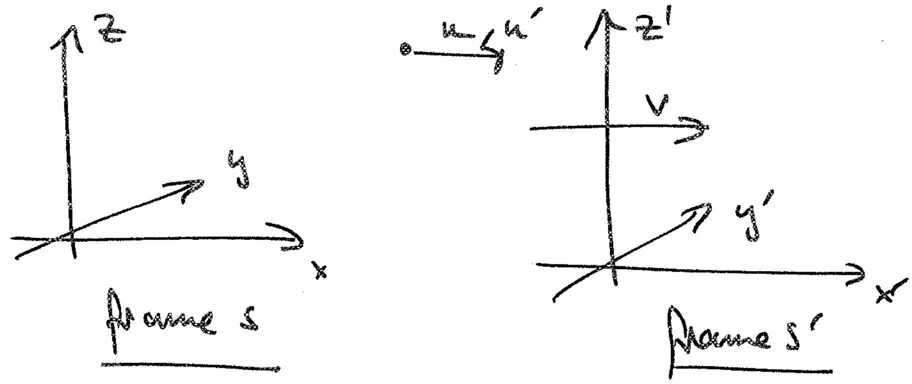
Special Relativity, Lecture 6: Velocity Transformations

Topics: Addition of velocities in classical (Newtonian) mechanics;
Velocity transformations in Special Relativity

Aim: To see how observers in different inertial frames of reference perceive the velocity of objects moving with constant velocity

6.1. Addition of velocities in classical (Newtonian) mechanics



particle velocity u in S $\qquad u' = u - v$

or in general, if velocities are not parallel to x-axis:

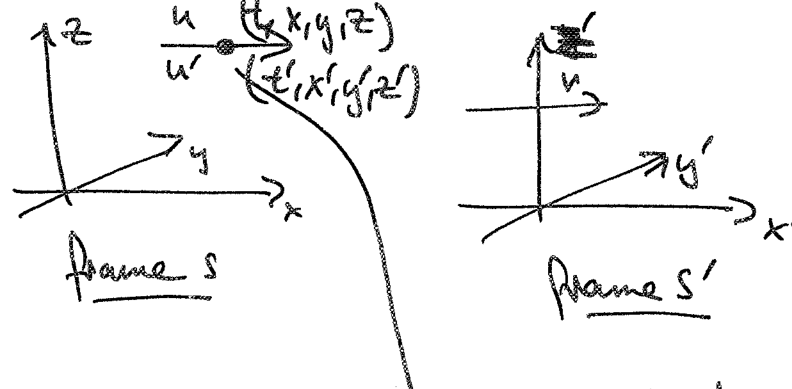
$$\vec{u}' = \vec{u} - \vec{v} \quad , \quad \vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} \ldots$$

This simple addition of velocities cannot be correct in special relativity.

Replacing the object by a light beam, we immediately have a contradiction with Einstein's second postulate (E2).

6.2 Velocity transformation in Special Relativity

Velocity is the rate of change of position with time. We will use the transformation of position and time (Lorentz transformation) to deduce the transformation of velocity.



moving object seen in both frames (first assume u parallel to x-axis)

$$u = u_x = \frac{dx}{dt} \qquad u' = u'_x = \frac{dx'}{dt'} \qquad (6.1)$$

These should be viewed/understood as the limits

$$u_x = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t}$$

$$u'_x = \lim_{\Delta t' \to 0} \frac{\Delta x'}{\Delta t'}$$

Lorentz transformation:

$$\Delta x' = \gamma(\overset{x_2 - x_1}{\Delta x} - v\Delta t)$$

$$\Delta t' = \gamma\left(-\frac{v}{c^2}\Delta x + \Delta t\right)$$

in the infinitesimal limit:

$$dx' = \gamma(dx - v\,dt)$$

$$dt' = \gamma\left(-\frac{v}{c^2}dx + dt\right) \qquad (6.2)$$

$$\boxed{u' = u'_x} = \frac{dx'}{dt'} \overset{(6.2)}{=} \frac{dx - v\,dt}{-\frac{v}{c^2}dx + dt} = \frac{\frac{dx}{dt} - v}{-\frac{v}{c^2}\frac{dx}{dt} + 1} = \boxed{\frac{u - v}{1 - \frac{uv}{c^2}}} \quad (6.3)$$

Note: For $uv \ll c^2$ (e.g. for a ball thrown with velocity u on Galileo's ship moving with v relative to coastline) the classical relation $u' = u - v$ is an excellent approximation of (6.3). Deviations would be too small to be measured.

Inversion of velocity transformation (6.3):

$$u' - \frac{u'v}{c^2} \cdot u = u - v$$

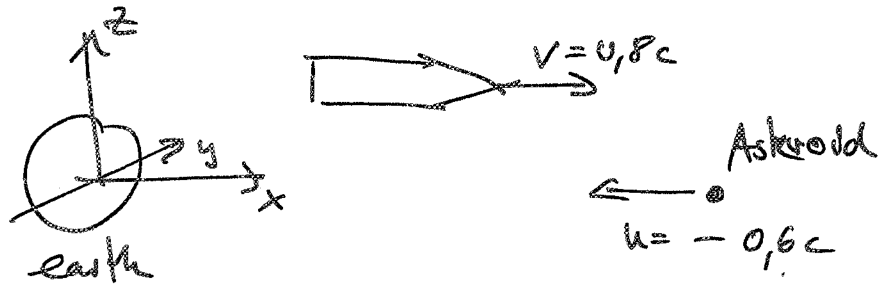
$$\Leftrightarrow \quad u' + v = \left(1 + \frac{u'v}{c^2}\right) u$$

$$\Leftrightarrow \quad \boxed{u = \frac{u'+v}{1+\frac{u'v}{c^2}}} \qquad (6.4)$$

As for the Lorentz transformation, the transformations from $S \to S'$ (6.3) and from $S' \to S$ (6.4) are related by $u \Leftrightarrow u'$ and $v \to -v$.

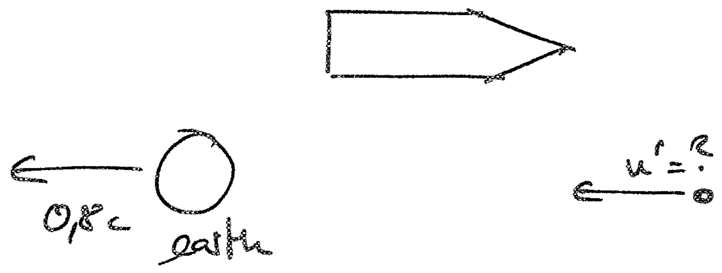
Example:

Frame S:



Q: How is this measured from the spaceship? What is the velocity u' of the Asteroid?

View from ship (frame S'):



$$v = 0{,}8c \; ; \; u = -0{,}6c$$

$$\Rightarrow \quad u' = \frac{u-v}{1-\frac{uv}{c^2}} = \frac{-1{,}4c}{1+\frac{0{,}8 \cdot 0{,}6\, c^2}{c^2}} = -0{,}946c$$

Note that classically, we would expect $u' = u - v = -1{,}4c$. Hence, the speed of the asteroid would be faster than the speed of light.

In special relativity this is not possible:

Proof:

Consider $|u| < c \; (\Leftrightarrow -c < u < c)$ and $v < c$

From (6.3) it follows that

$$u' - c = \frac{u-v}{1-\frac{vu}{c^2}} - c = \frac{-\overbrace{(c-u)}^{>0}\,\overbrace{(c+v)}^{>0}}{c\underbrace{\left(1-\frac{vu}{c^2}\right)}_{>0}} < 0$$

$$\Leftrightarrow \quad u' < c \quad (1)$$

$$u' + c = \frac{u-v}{1-\frac{vu}{c^2}} + c = \frac{\overbrace{(c+u)}^{>0}\,\overbrace{(c-v)}^{>0}}{c\underbrace{\left(1-\frac{vu}{c^2}\right)}_{>0}} > 0$$

$$\Leftrightarrow \quad u' > -c \quad (2)$$

From (1) and (2) it follows that

$$-c < u' < c \quad \Leftrightarrow \quad |u'| < c$$

In no frame the speed of the object exceeds the speed of light!

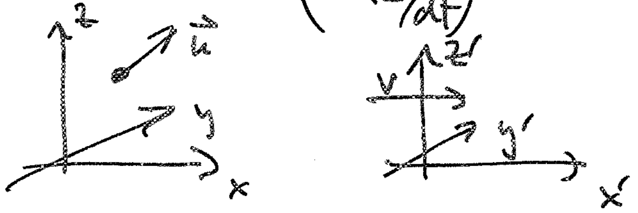
Q: Is the velocity transformation consistent with the constancy of the speed of light (E2)?

For $u = c$ we should get $u' = c$ according to Einstein's second postulate

$$u = c \Rightarrow u' = \frac{c - v}{1 - \frac{vc}{c^2}} = \frac{c-v}{1 - \frac{v}{c}} = c\,\frac{c-v}{c-v} = c \quad \checkmark$$

6.3 Transformation of velocities not parallel to x-axis

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} dx/dt \\ dy/dt \\ dz/dt \end{pmatrix} \qquad \vec{u}' = \begin{pmatrix} u_x' \\ u_y' \\ u_z' \end{pmatrix} = \begin{pmatrix} dx'/dt' \\ dy'/dt' \\ dz'/dt' \end{pmatrix}$$



$$dx' = \gamma(dx - v\,dt)$$
$$dt' = \gamma\left(-\frac{v}{c^2}dx + dt\right)$$

$$\left.\begin{aligned} dy' &= dy \\ dz' &= dz \end{aligned}\right\}$$ since v parallel to x axis, y and z coordinates are not changed under Lorentz transformation

Transformation of x-component of velocity exactly as before:

$$\boxed{u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}}$$

Transformation of y-component:

$$u_y' = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(-\frac{v}{c^2}dx + dt\right)}$$

$$= \frac{dy/dt}{\gamma\left(-\frac{v}{c^2}\frac{dx}{dt} + 1\right)}$$

$$= \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

The z-component transforms analogously:

$$u_z' = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

Summarize:

$$\vec{u}' = \begin{pmatrix} u_x' \\ u_y' \\ u_z' \end{pmatrix} = \frac{1}{1 - \frac{v u_x}{c^2}} \begin{pmatrix} u_x - v \\ u_y/\gamma \\ u_z/\gamma \end{pmatrix} \qquad (6.4)$$

inversion:

$$\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \frac{1}{1 + \frac{v u_x'}{c^2}} \begin{pmatrix} u_x' + v \\ u_y'/\gamma \\ u_z'/\gamma \end{pmatrix}$$