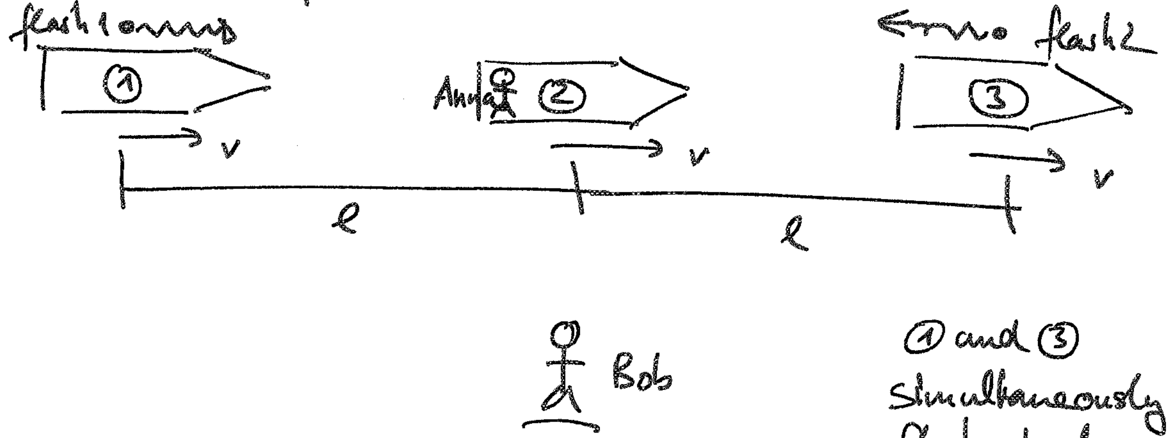
Special Relativity, Lecture 4: Consequences of Lorentz Transformations

Topics: Simultaneity; Time Dilation; Length Contraction; Some Examples; The Invariant Interval

Aim: Putting Lecture 2 on a more mathematical footing by using the Lorentz transformation

4.1. Simultaneity

Return to the example used in Lecture 2



① and ③ simultaneously flash torches at Anna

In frame S' (Anna):

flash 1 : $(t_1', x_1') = (0, 0)$
flash 2 : $(t_2', x_2') = (0, 2l)$

events simultaneous in Anna's frame : $\Delta t' = t_2 - t_1 = 0$

In frame S (Bob):

Use Lorentz transformation to transform the coordinates of the two events.

S' is moving relative to S with velocity v

$$\rightarrow \begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v/c \\ -\gamma v/c & \gamma \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

invert transformation : $(t, x) \leftrightarrow (t', x')$ and $v \rightarrow -v$

$$\begin{pmatrix} ct \\ x \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v/c \\ \gamma v/c & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ x' \end{pmatrix}$$

$$\Rightarrow ct = \gamma ct' + \gamma \frac{v}{c} x' \Rightarrow t = \gamma t' + \gamma \frac{v}{c^2} x'$$

flash 1 : $t_1 = \gamma(t_1' + \frac{v}{c^2} x_1') = 0$

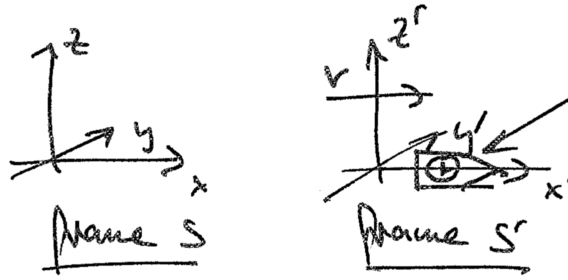
flash 2 : $t_2 = \gamma(t_2' + \frac{v}{c^2} x_2') = \gamma \frac{v}{c^2} 2l$

$$\Rightarrow \quad \Delta t = t_2 - t_1 = \gamma \frac{v}{c^2} 2l > 0$$

The two events which are simultaneous in Anna's frame (but not at the same location) are not simultaneous in Bob's frame.

4.2. Time Dilation

Proper time: The time between two events in the inertial frame of reference where they occur at the same location.



clock at rest in S', always at the same location in S' (proper time of observer in spaceship)

two events:
event 1: $(t_1', x_1') = (t_1', x')$
event 2: $(t_2', x_2') = (t_2', x')$
$\Delta t' = t_2' - t_1'$

Frame S: Lorentz transformation

event 1: $t_1 = \gamma(t_1' + \frac{v}{c^2} x_1') = \gamma t_1' + \gamma \frac{v}{c^2} x'$

event 2: $t_2 = \gamma(t_2' + \frac{v}{c^2} x_2') = \gamma t_2' + \gamma \frac{v}{c^2} x'$

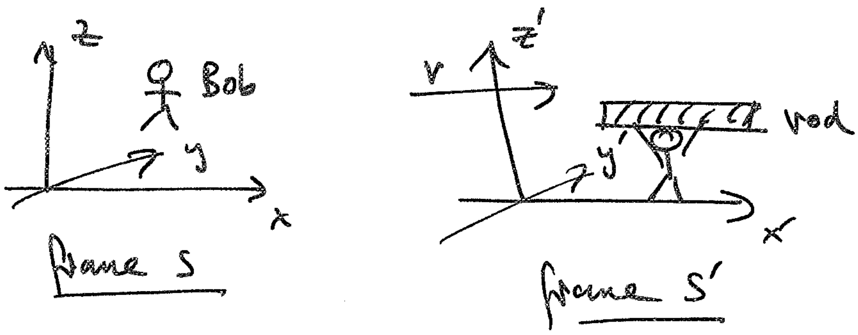
$$\Rightarrow \quad \Delta t = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma \Delta t'$$

Note:

- $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \geq 1$ since $v \leq c$
- $\Delta t \geq \Delta t'$ (time dilation)
- proper time is the minimum ~~spati~~ temporal separation between events

4.3. Length Contraction

Proper Length (Rest Length): The length l_0 of an object in a frame in which it is at rest.



To measure the length of the rod, Bob simultaneously measures the positions of the two ends

frame S: event 1: $(t_1, x_1) = (t, x_1)$
event 2: $(t_2, x_2) = (t, x_2)$

length $\Delta x = x_2 - x_1$

frame S': rest frame of Anna and the rod

event 1: (t_1', x_1')

event 2: (t_2', x_2')

length $\Delta x' = x_2' - x_1' = l_0$ (proper length)

Lorentz transformation:

$$x_1' = \gamma(x_1 - vt_1) = \gamma x_1 - \gamma vt$$
$$x_2' = \gamma(x_2 - vt_2) = \gamma x_2 - \gamma vt$$

$$\Rightarrow \Delta x' = l_0 = \gamma \Delta x$$

$$\Rightarrow \Delta x = l_0/\gamma$$

Note:
- Since $\gamma \geq 1$: $\Delta x \leq l_0$ (length contraction)
- Proper length is the maximum spatial separation of two events (or maximum length of an object)

4.4. Some Examples

Aim:
- Straightforward application of length contraction and time dilation
- Show how apparent contradictions can be resolved by looking at a problem from different points of view

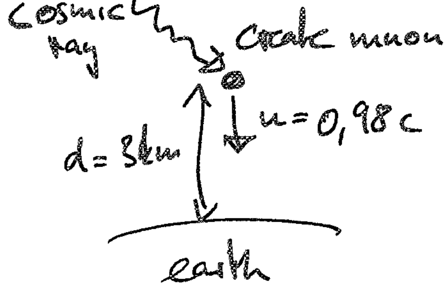
Example 1: Muon decay in the atmosphere

- Muons are subatomic particles created by cosmic rays in the upper atmosphere
- unstable to decay into positrons an neutrinos:

$$\underset{\text{muon}}{\mu^+} \longrightarrow \underset{\text{positron}}{e^+} + \underbrace{\nu_e + \bar{\nu}_\mu}_{\text{neutrinos}}$$

At rest, the average lifetime of a muon is $\tau = 2{,}2\,\mu s = 2{,}2 \cdot 10^{-6}\,s$

- Given the altitude that muons are produced at and the speed with which they travel, they should not be able to reach the earth, but they do !!



1) Classically, how long would it take the muon to reach the earth?

$$t = \frac{d}{u} = \frac{3\,km}{0{,}98\,c} \simeq \frac{3\,km}{0{,}98 \cdot 3 \cdot 10^5 \frac{km}{s}} = 10{,}2 \cdot 10^{-6}\,s = 10{,}2\,\mu s$$

This time is much longer than the average lifetime. Therefore the muon is expected to decay before it would reach the earth.

Q: Why do muons reach the earth?

A: The classical picture is insufficient! The speed of the muon ($u = 0{,}98c$) is close to the speed of light. Therefore, relativistic effects are very important.

2) Time Dilation and view from earth

The lifetime of the muon should be measured in a frame where the muon is at rest (in the muon proper time). Therefore imagine a clock attached to the muon.

Because of time dilation an observer on the earth measures a longer lifetime

$$\underset{\substack{\uparrow \\ \text{earthbound} \\ \text{clock}}}{\Delta t} = \gamma \underset{\substack{\uparrow \\ \text{muon clock}}}{\Delta t_0} \quad ; \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - 0{,}98^2}} \simeq 5{,}03$$

$$\Rightarrow \Delta t = 5{,}03 \cdot 2{,}2\,\mu s = 11{,}07\,\mu s$$

This time is longer than the time it would take to travel to the earth, so muon would reach!

3) Length contraction and view from muon

How does the muon see this?

An earthbound observer sees the muon travel a distance of $l_0 = 3\,km$ (proper length)

The muon sees the distance length contracted:

$$l = l_0/\gamma = \frac{3\,km}{5{,}03} \simeq 596\,m$$

From 1st postulate, the muon sees the earth approaching at $u = 0{,}98c$ from a distance of $l = 596\,m$

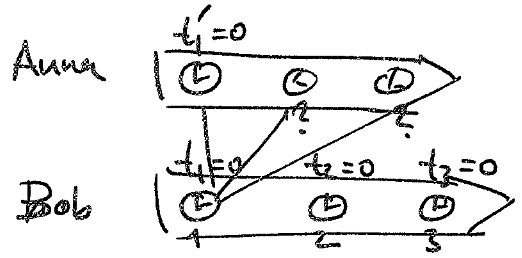
$$\Rightarrow \text{time till collision: } \Delta t = \frac{l}{u} = \frac{596\,m}{0{,}98c} = 2{,}03\,\mu s$$

Shorter than lifetime of the muon, so muon would reach

Note:
- The fact that the muon reaches the earth is an event agreed upon by all observers
- They have different views of the details of how this happens
- Example shows the interdependence of length contraction and time dilation

Example 2 : Synchronization

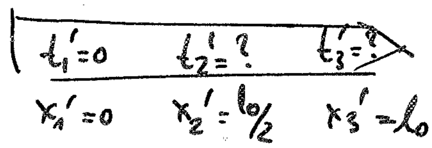
- Anna and Bob have two identical spaceships, they both synchronize the clocks on their ships
- Anna passes Bob with relative velocity v
- When tails of ships are next to each other, Bob (frame S) reads $t=0$ on all of his clocks and $t'=0$ on the clock on the tail of Anna's ship



(Anna's ship looks shorter from Bob's point of view)

Q: What does Bob read on Anna's clocks in the center and front of her rocket?

A: In Anna's frame (S') the three events are



l_0 : proper length of the spaceship

Transforming the coordinates back to Bob's frame by using the Lorentz transformation:

$$0 = t_1 = \gamma\left(\frac{v}{c^2}x_1' + t_1'\right) = \gamma\left(\frac{v}{c^2}\cdot 0 + 0\right) = 0 \quad \checkmark \text{ agrees}$$

$$0 = t_2 = \gamma\left(\frac{v}{c^2}x_2' + t_2'\right) = \gamma\left(\frac{v}{c^2}\frac{l_0}{2} + t_2'\right) \Rightarrow t_2' = -\cancel{\gamma}\frac{v}{c^2}\frac{l_0}{2}$$

$$0 = t_3 = \gamma\left(\frac{v}{c^2}x_3' + t_3'\right) = \gamma\left(\frac{v}{c^2}l_0 + t_3'\right) \Rightarrow t_3' = -\cancel{\gamma}\frac{v}{c^2}l_0$$

Anna's clocks appear not to be synchronized as measured by Bob.

4.5 The invariant interval

We have spent a lot of time thinking about the differences between different frames of reference. However, there are important things that are the same for all observers.

1) Events: events that are unique in one inertial frame of reference are unique in all

2) Causality: What causes what is preserved by the Lorentz transformation

The latter can be seen by constructing the invariant interval I:

Consider 2 events separated by Δt and $\Delta \underline{r}$ in a frame S and by $\Delta t'$ and $\Delta \underline{r}'$ in S' moving with velocity v relative to S
(Without loss of generality we again assume v parallel to x direction)

Claim: $$I \,(=s^2) = c^2 \Delta t^2 - \Delta \underline{r}^2 \overset{!}{=} c^2 \Delta t'^2 - \Delta \underline{r}'^2$$

(sometimes the interval is denoted by s^2)

Proof: Since $y' = y$ and $z' = z$, claim is equivalent to
$$c^2 \Delta t^2 - \Delta x^2 = c^2 \Delta t'^2 - \Delta x'^2$$

Lorentz trf.: $$\begin{cases} t' = \gamma t - \gamma \frac{v}{c^2} x \\ x' = -\gamma v t + \gamma x \end{cases} \Longrightarrow \begin{cases} \Delta t' = \gamma \Delta t - \gamma \frac{v}{c^2} \Delta x \\ \Delta x' = -\gamma v \Delta t + \gamma \Delta x \end{cases}$$

$\Delta t = t_2 - t_1$
$\Delta t' = t_2' - t_1'$
...

$$c^2 \Delta t'^2 - \Delta x'^2 = c^2 \left(\gamma \Delta t - \gamma \frac{v}{c^2} \Delta x\right)^2 - \left(-\gamma v \Delta t + \gamma \Delta x\right)^2$$
$$= c^2\left(\gamma^2 \Delta t^2 - 2\gamma^2 \frac{v}{c^2} \Delta t \Delta x + \gamma^2 \frac{v^2}{c^4} \Delta x^2\right)$$
$$- \left(\gamma^2 v^2 \Delta t^2 - 2\gamma^2 v \Delta t \Delta x + \gamma^2 \Delta x^2\right)$$

$$= \gamma^2(c^2-v^2)\Delta t^2 + \gamma^2\left(\frac{v^2}{c^2}-1\right)\Delta x^2$$

$$= c^2\Delta t^2 - \Delta x^2$$

$$\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}}$$

Pairs of events may be separated into 3 types, those with $I>0$, $I<0$ and $I=0$

Time like interval ($I>0$):

When $I>0$, there exists an inertial frame in which the events occur at the same location but different times. Such events are causally connected: one could have caused the other

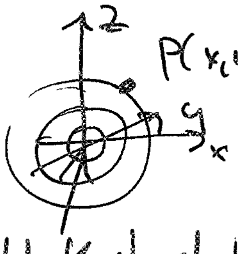
Space like intervals ($I<0$):

When $I<0$, there exists an inertial frame of reference in which the events occur at the same time but at different locations.
Such events cannot be causally connected since information cannot travel faster than the speed of light. One event could not have caused the other.

Light like interval ($I=0$).

In this special case we have $c^2\Delta t^2 = \Delta \underline{r}^2$

Note that we encountered this relation when we introduced a recipe for clock synchronization making use of the constancy of the speed of light



light flash at $t=0$
(event 1 $(t,\underline{r})=(0,\underline{0})$)

event 2: light reaches P
set clock in P to $t=\frac{d}{c}$

$$\Rightarrow c^2t^2 = d^2 = x^2+y^2+z^2 = \underline{r}^2$$