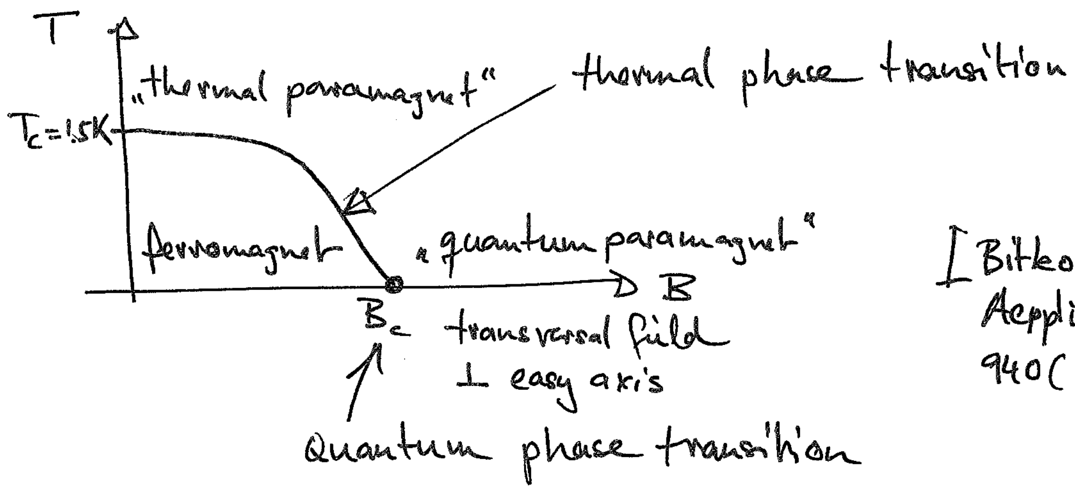
1. Introduction

1.1. What are quantum-phase transitions

- Happen at $T=0$
- Continuous or first order
  Here we focus on continuous QPTs
- Driven by quantum fluctuations since there are no thermal fluctuations at $T=0$
- Tuning by control parameter like pressure, magnetic field, chemical doping

<u>Example</u>: $LiHoF_4$



[Bitko/Rosenbaum/Aeppli, Phys. Rev. Lett. **77**, 940 (1996)]

<u>Theoretical model description</u>

→ Quantum Hamiltonian $H = H_0 + g H_1$

g ← dimensionless tuning parameter

- If H_0 and H_1 favor different ground states we expect a quantum phase transition at a critical value $g=g_c$
- usually $[H_0, H_1] \neq 0$ ⇒ no simultaneous diagonalization

1) Previous example : Ising model in a transversal field

- $LiHoF_4$ ionic crystal
- at low T, only degrees of freedom are the spins of the Holmium atoms
- easy axis : spins prefer to point up or down with respect to a certain crystal axis

$$H_0 = -J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z \qquad \text{where } \hat{S}_i^z = \frac{\hbar}{2} \hat{\sigma}_i^z$$

sum over nearest neighbor bonds

Hamiltonian is diagonal in the simultaneous eigenbasis of $\hat{S}^2$ and $\hat{S}^z$: $|s, m_s\rangle = |1/2, m_s\rangle$, $m_s = \pm 1/2$

$|1/2, 1/2\rangle =: |\uparrow\rangle$, $|1/2, -1/2\rangle =: |\downarrow\rangle$

$$\left.\begin{array}{l} \hat{S}_i^z |\uparrow\rangle_i = \frac{\hbar}{2} |\uparrow\rangle_i \\ \hat{S}_i^z |\downarrow\rangle_i = -\frac{\hbar}{2} |\downarrow\rangle_i \end{array}\right\} \longrightarrow \left\{\begin{array}{l} \hat{\sigma}_i^z |\uparrow\rangle = |\uparrow\rangle \\ \hat{\sigma}_i^z |\downarrow\rangle = -|\downarrow\rangle \end{array}\right.$$

→ matrix form $\underline{\underline{\sigma}}^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- Energy $E_0(\{m_i\}) = -J \sum_{\langle ij \rangle} m_i m_j$ with $m_i = \pm 1$

→ 2 degenerate groundstates : $|0\rangle_1 = \prod_i |\uparrow\rangle_i$, $|0\rangle_2 = \prod_i |\downarrow\rangle_i$

- Thermodynamic system will spontaneously pick one of the two states
  → spontaneous breaking of $\mathbb{Z}_2$ symmetry
- transversal magnetic field perpendicular to the easy axis : $H_1 = -h \sum_i \hat{\sigma}_i^x$

$\hat{\sigma}_i^x |\uparrow\rangle = |\downarrow\rangle$, $\hat{\sigma}_i^x |\downarrow\rangle = |\uparrow\rangle$ → matrix form $\underline{\underline{\sigma}}^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

H_1 favors non-degenerate ground state $|0\rangle = \prod_i |\rightarrow\rangle_i$

$|\rightarrow\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$

Ising model in a transversal field :

$$\boxed{H = -J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - gJ \sum_i \hat{\sigma}_i^x} \qquad (h = gJ)$$

- what happens if we gradually increase g?

  transversal field induces spin flips: $\hat{\sigma}^x|\uparrow\rangle = |\downarrow\rangle$, $\hat{\sigma}^x|\downarrow\rangle = |\uparrow\rangle$

  and therefore destabilizes the magnetic order.

- Magnetization will decrease as we increase g and eventually vanish as $g \nearrow g_c$

- We will discuss the quantum-phase transition in this model in detail in Chapter 2 in the case $d=1$

2) Quantum antiferromagnet ($\to$ Chapter 4)

$$\mathcal{H}_0 = J \sum_{\langle ij\rangle} \hat{\underline{S}}_i \cdot \hat{\underline{S}}_j \quad (J>0), \quad \hat{\underline{S}}_i = \begin{pmatrix} \hat{S}_i^x \\ \hat{S}_i^y \\ \hat{S}_i^z \end{pmatrix}$$

$\uparrow \ \downarrow \ \uparrow$

$\downarrow \ \uparrow \ \downarrow$

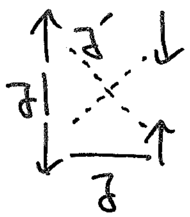
site $i \hat{=} (x,y)$

order parameter: staggered magnetization

$$\vec{m} = (-1)^{x+y} \langle \hat{\underline{S}}_i \rangle$$

- spontaneous breaking of spin-rotation symmetry
- At the Neel temperature T_N, long range magnetic order disappears (thermal phase transition)
- Can we induce a quantum phase transition?

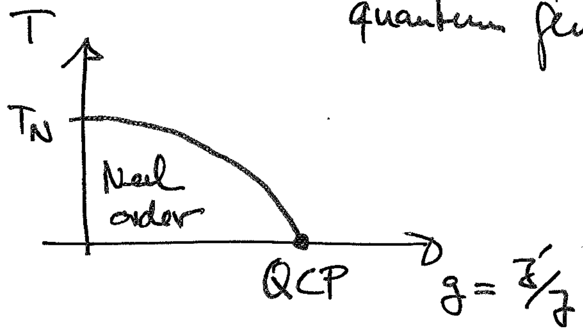
  yes, e.g. if we turn on antiferromagnetic next-nearest neighbor couplings J'



$J>0$
$J'>0$

$$\boxed{\mathcal{H} = J \sum_{\langle ij\rangle} \hat{S}_i \hat{S}_j + \underset{\underset{gJ}{\parallel}}{J'} \sum_{\langle\langle ij\rangle\rangle} \hat{S}_i \hat{S}_j}$$

J' induces <u>frustration</u> and increases quantum fluctuations



Note: • Although this model is relevant to many real systems, it is difficult to tune the parameter g experimentally

3) Bose - Hubbard model (Chapter 3)

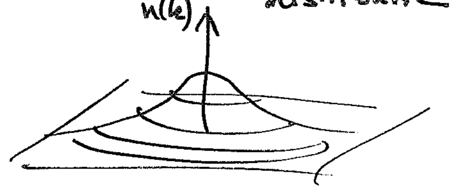
Bosons hopping on a d-dim. lattice with amplitudes t between neighboring sites, chemical potential μ, on-site repulsion U:

$$\boxed{H = -t \sum_{\langle ij \rangle} \left(b_i^\dagger b_j + h.c. \right) - \mu \sum_i \underbrace{\hat{n}_i}_{= b_i^\dagger b_i} + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1)}$$

- $\underline{U \gg t}$: Strong interactions $\rightarrow$ Mott insulator (MI)

particles become effectively distinguishable

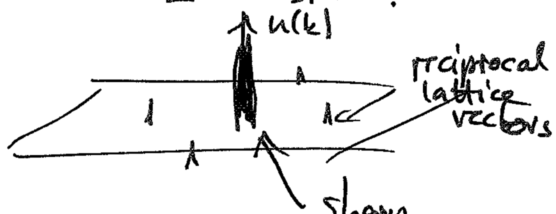
momentum space: broad distribution



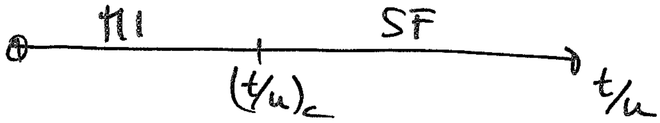
- $\underline{U \ll t}$: Weak interactions $\rightarrow$ Superfluid (SF)

real-space picture no longer appropriate since bosons are indistinguishable

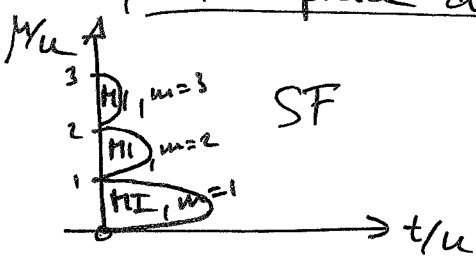
all bosons can condense in the $\underline{k} = 0$ state!



- Quantum phase transition



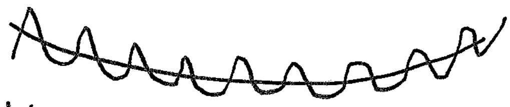
- The location of the transition depends on the filling and therefore the chemical potential μ
  $\rightarrow$ quantum phase diagram



- Mott lobes with different numbers of bosons per site, $n=2$ 
- At $t=0$: Minimize $E(n) = -\mu n + \frac{U}{2} n(n-1)$ for integer n

- Transition observed in optical lattice experiments

experimental problems:
- very small systems, far from the thermodynamic limit
- trap potential → spatial inhomogeneity



- heating by the lasers
  → T is not small

1.2. Basic concepts of phase transitions and critical behavior

- continuous phase transitions can usually be characterized by an order parameter, e.g. the magnetization in an Ising ferromagnet: $m = \langle \hat{\sigma}_i^z \rangle$
- In the ordered phase, $m \neq 0$, and the correlation function shows long range order

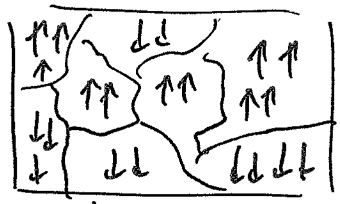
$$G_{ij} = G(|\underline{r}_i - \underline{r}_j|) = \langle \hat{\sigma}_i^z \hat{\sigma}_j^z \rangle \xrightarrow[|\underline{r}_i - \underline{r}_j| \to \infty]{} m^2$$

- In the disordered phase, thermodynamic average of the order parameter is zero, $m = \langle \hat{\sigma}_i^z \rangle = 0$

Fluctuations are non-zero. Spin will be correlated up to a length scale ξ (correlation length)

$$G(|\underline{r}_i - \underline{r}_j|) \sim e^{-|\underline{r}_i - \underline{r}_j|/\xi}$$

The order is short ranged.



patches of typical size ξ

- As we approach the critical point, the correlation length has to diverge

$$\xi \underset{t \to 0_+}{\sim} |t|^{-\nu} \qquad \text{where } t = \frac{T - T_c}{T_c} \text{ for thermal phase transition}$$

ν correlation-length exponent

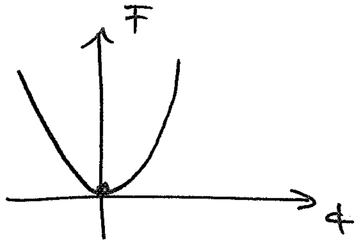
$$\left(t = \frac{g - g_c}{g_c} \text{ at } T = 0, \text{ QPT} \right)$$

- close to the critical point, there is no other scale than ξ, at the critical point the system becomes scale invariant
- If ξ is the only scale and $\xi \sim |t|^{-\nu}$ as $t \to 0_+$, the transition, or better, the nature of the transition should be independent of microscopic details
- Concept of universality: Nature of transitions depend only on
  a) dimensionality
  b) Symmetry of the order parameter
- Nature of transition can be understood in terms of an effective long-wavelength description (coarse graining, continuum limit, gradient expansion)
  → Landau functional of free energy in terms of order parameter field $\vec{\phi}(\underline{r})$ (e.g. the local magnetization $\vec{m}(\underline{r})$ in a magnet)

$$F[\vec{\phi}] = \int d^d\underline{r} \left\{ |\nabla\vec{\phi}|^2 + t|\vec{\phi}|^2 + u|\vec{\phi}|^4 - \vec{B}\cdot\vec{\phi} \right\}$$

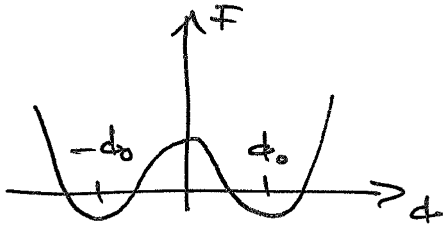
- $\vec{\phi}$ order parameter with N components
- For Ising model: $N=1$ since direction of magnetization is fixed
- In the lattice model (zero field) we have the symmetry $m_i \to -m_i$ on ~~for~~ all sites (the Hamiltonian is invariant under this transformation)
- The Landau functional $F[\phi] = \int d^d r \{(\nabla\phi)^2 + t\phi^2 + u\phi^4\}$ preserves this $\mathbb{Z}_2$ symmetry (for $\vec{B}=0$): $F[\phi]$ is unchanged if we replace $\phi(r)$ by $-\phi(r)$

$t>0$:



$\phi_0 = 0$: paramagnet

$t<0$:



$m = \pm\phi_0$ two minima, spontaneous breaking of $\mathbb{Z}_2$ symmetry

1.3. Critical exponents, scaling and hyperscaling

$\boxed{t = \frac{T - T_c}{T_c}}$

	exponent	definition	Conditions
Specific heat $C = -T \frac{\partial^2 F}{\partial T^2}$	α	$C \sim \lvert t\rvert^{-\alpha}$	$t \to 0, B = 0$
order parameter	β	$m \sim \lvert t\rvert^{\beta}$	$t \to 0_-, B = 0$
Susceptibility $\chi = -\frac{\partial^2 F}{\partial B^2}\Big\rvert_{B=0}$	γ	$\chi \sim \lvert t\rvert^{-\gamma}$	$t \to 0, B = 0$
critical isotherm	δ	$m \sim B^{1/\delta}$	$t = 0, B \to 0$
correlation length	ν	$\xi \sim \lvert t\rvert^{-\nu}$	$t \to 0_+, B = 0$
Correlation function	η	$G(r) \sim r^{-d+2-\eta}$	$t = 0, B = 0$

- Set of critical exponents characterizes the nature of the phase transition
- Exponents $\alpha, \beta, \gamma, \delta, \nu, \eta$ are not independent from each other but related by <u>scaling</u> and sometimes <u>hyperscaling</u> relations
- Note: This β is not to be confused with inverse temperature!

- Close to transition, ξ is the only relevant length scale

  $\Rightarrow$ Physical properties must be unchanged if we rescale all length by a common factor b and at the same time rescale the parameter such that ξ retains its old value

  $\Rightarrow$ free energy density must be a homogeneous function:

$$f(t,B) = b^{-d} f(b^r t, b^s B)$$

- It follows immediately that f can be written in the form

$$\boxed{f(t,B) = |t|^{d/r} \, \psi_\pm\left(\frac{B}{|t|^{s/r}}\right)}$$

$\psi_\pm$ scaling function, usually diffrent for $t>0$ (ψ_+) and $t<0$ (ψ_-)

order parameter: $(t<0)$

$$m = -\frac{\partial f}{\partial B}\Big|_{B=0} = -|t|^{d/r} \cdot \frac{1}{|t|^{s/r}} \psi_-'(0)$$

$$\sim |t|^{\frac{d-s}{r}} \overset{!}{=} |t|^\beta$$

$$\Rightarrow \beta = \frac{d-s}{r} \quad (1)$$

susceptibility:

$$\chi = -\frac{\partial^2 f}{\partial B^2}\Big|_{B=0} = -|t|^{\frac{d-2s}{r}} \psi_\pm''(0) \overset{!}{=} |t|^{-\gamma}$$

$$\Rightarrow \gamma = -\frac{d-2s}{r} \quad (2)$$

Specific heat:

$$C = -T \frac{\partial^2 f}{\partial T^2}\Big|_{B=0} = -\frac{T}{T_c^2} \frac{\partial^2 f}{\partial t^2}\Big|_{B=0}$$

$$= \frac{d}{r}\left(\frac{d}{r}-1\right) |t|^{d/r-2} \psi_\pm(0) \sim |t|^{d/r-2} \overset{!}{=} |t|^{-\alpha}$$

$$\Rightarrow \alpha = 2 - \frac{d}{r} \quad (3)$$

it follows that $\boxed{\alpha + 2\beta + \gamma = 2}$ Rushbrooke's scaling law

Critical isotherm: $m \sim B^{1/\delta}$ as $B \to 0$ and $t = 0$

we rewrite the homogeneity relation:

$$f(t,B) = |t|^{d/r}\, \Psi_\pm\left(\frac{B}{|t|^{s/r}}\right)$$

$$= \underbrace{|t|^{d/r}\left(\frac{B}{|t|^{s/r}}\right)^{d/s}}_{\text{to get rid of } t \text{ here}} \tilde{\Psi}_\pm\left(\frac{B}{|t|^{s/r}}\right)$$

$$= B^{d/s}\, \tilde{\Psi}_\pm\left(\frac{B}{|t|^{s/r}}\right)$$

$$m = -\frac{\partial f}{\partial B} = \underset{\substack{\uparrow \\ \text{leading term for } B \to 0}}{-\frac{d}{s} B^{d/s-1}}\, \tilde{\Psi}_\pm\left(\frac{B}{|t|^{s/r}}\right) - B^{d/s}\, \tilde{\Psi}'\left(\frac{B}{|t|^{s/r}}\right) \frac{1}{|t|^{s/r}}$$

$$\underset{t \to 0}{\approx} -\frac{d}{s} B^{d/s-1}\, \tilde{\Psi}_\pm(\infty)$$

$$\Rightarrow \quad 1/\delta = \frac{d}{s} - 1 \quad \Big/ \cdot s/r \qquad \underbrace{(1 + 1/\delta)\, s/r}_{\substack{=\gamma+\beta \\ = 2-\alpha-\beta}} = \underbrace{\frac{d}{r}}_{2-\alpha}$$

$$\Rightarrow \quad \boxed{\alpha + \beta(\delta+1) = 2} \quad \text{Griffith's scaling law}$$

Two point correlation function:

$$G(r) = \langle (m(r) - m)(m(0) - m) \rangle$$

$$= \langle m(r)\, m(0) \rangle - m^2$$

If we define $G(r)$ like this (subshracting $m^2 = \begin{cases} 0 & t>0 \\ c \neq 0 & t<0 \end{cases}$),

$G(r) \xrightarrow[r \to \infty]{} 0$ for $t<0$ and $t>0$

In general, $G(r)$ has to be of the form

$$G(r) = \frac{\Phi_\pm(r/\xi)}{r^{d-2+\eta}}$$

only possible dimensionless combination

Example: $(t>0)$

$$F = \int d^d r \left\{ (\nabla \vec{m})^2 + t\, \vec{m}^2 \right\} = \int \frac{d^d q}{(2\pi)^d} \, (q^2+t)\, \vec{m}(\underline{q})\, \vec{m}(-\underline{q})$$

$$\langle \vec{m}(\underline{q})\, \vec{m}(\underline{q}') \rangle = \underbrace{G(q)}_{=(q^2+t)^{-1}}\, \delta(\underline{q}+\underline{q}')$$

[Derive this by adding a source term $\int \frac{d^d q}{(2\pi)^d} \vec{J}(\underline{q})\, \vec{m}(-\underline{q})$

$$\left.\frac{\delta^2 Z_J}{\delta \vec{J}(\underline{q})\, \delta \vec{J}(-\underline{q}')}\right|_{\vec{J}=0} = \int \mathcal{D}[\vec{m}]\; \vec{m}(\underline{q})\, \vec{m}(\underline{q}')\, e^{-\int \frac{d^d q}{(2\pi)^d} (q^2+t)\, \vec{m}(\underline{q})\, \vec{m}(-\underline{q})}$$

$$= Z \langle \vec{m}(\underline{q})\, \vec{m}(\underline{q}') \rangle$$

Now calculate the left-hand side explicitly by completing the square in the Gaussian integral]

Calculate the Fourier transform of $G(q) = \frac{1}{q^2+t}$:

$d=3$:
$$G(r) = \int \frac{d^3 q}{(2\pi)^3} \frac{e^{i\underline{q}\underline{r}}}{q^2+t}$$

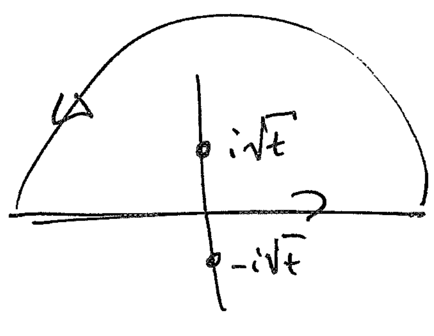
$$= \frac{1}{(2\pi)^3} \int_0^{2\pi} d\varphi \int_0^\infty dq\, q^2 \int_0^\pi \sin\theta\, d\theta\, \frac{e^{iqr\cos\theta}}{q^2+t}$$

$$= \frac{1}{(2\pi)^2} \int_0^\infty dq\, \frac{q^2}{q^2+t} \underbrace{\int_{-1}^{1} d\sigma\, e^{iqr\sigma}}_{= \frac{e^{iqr}-e^{-iqr}}{iqr}}$$

$\sigma = \cos\theta$
$d\sigma = -\sin\theta\, d\theta$

$$= \frac{1}{i(2\pi)^2 r} \int_0^\infty dq\, \frac{q\,(e^{iqr}-e^{-iqr})}{q^2+t}$$

$$= \frac{1}{i(2\pi)^2 r} \int_{-\infty}^{\infty} dq\, \underbrace{\frac{q\, e^{iqr}}{q^2+t}}_{\frac{q\, e^{iqr}}{(q+i\sqrt{t})(q-i\sqrt{t})}}$$



$$= \frac{1}{i(2\pi)^2 r}\, 2\pi i\, \mathrm{Res}(i\sqrt{t}) = \frac{1}{2\pi r} \frac{i\sqrt{t}\, e^{-r\sqrt{t}}}{2i\sqrt{t}} = \boxed{\frac{e^{-r\sqrt{t}}}{4\pi r}}$$

$\Rightarrow \quad \xi = |t|^{-1/2} \rightarrow \underline{\nu = 1/2}$ and $\underline{\eta = 0}$

Relation of ν and η to other exponents?

- non-local magnetic susceptibility $\chi(\underline{r}-\underline{r}') = \left.\frac{\partial m(r)}{\partial B(r')}\right|_{B=0}$
- general theory of linear response implies that
$$\chi(\underline{q}) = G(\underline{q})$$
- Uniform susceptibility:
$$\chi = \chi(\underline{q}=0) = G(\underline{q}=0) = \int d^d r \; G(\underline{r})$$
$$= \int d^d r \; \frac{\Phi_\pm(r/\xi)}{r^{d-2+\eta}}$$
$$\underset{z=r/\xi}{=} \xi^{2-\eta} \int d^d z \; \frac{\Phi_\pm(z)}{z^{d-2+\eta}}$$

Integral converges in both cases:

$t \geq 0$: Φ_+ exponentially small for large arguments (SRO)

$t \leq 0$: Φ_- exponentially small since deviations from finite magnetization are uncorrelated at large distances

$$\rightarrow \chi \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)} \overset{!}{=} |t|^{-\gamma}$$
$$\rightarrow \boxed{\gamma = \nu(2-\eta)} \quad \text{Fisher's scaling law}$$

Hyperscaling:

Since ξ is the only lengthscale close to T_c which should be relevant, it is reasonable to assume that the free energy density scales as
$$f \sim \xi^{-d} \sim |t|^{d\nu}$$

Differentiating twice with respect to temperature:
$$C \sim |t|^{d\nu-2} \overset{!}{=} |t|^{-\alpha}$$
$$\rightarrow \boxed{\alpha = 2 - d\nu} \quad \text{Josephson's scaling law}$$

Note: - Only scaling law that explicitly involves dimensionality of the system
- Additional scaling assumption (hyperscaling $f \sim \xi^{-d}$)
- In contrast to other 3 scaling laws, hyperscaling is not always satisfied

- set of critical exponents completely characterizes the critical behavior of a particular phase transition
- universality: critical exponents are the same for entire classes of phase transitions which occur in very different systems
- Universality classes are determined by
  a) symmetries of the Hamiltonian
  b) spatial dimensionality
- reason for universality: divergence of correlation length

  Close to transition, system averages over large distances rendering microscopic details of the Hamiltonian unimportant
- critical behavior is crucially determined by the relevance or irrelevance of order-parameter fluctuations
- fluctuations become increasingly important as the dimensionality of the system is reduced

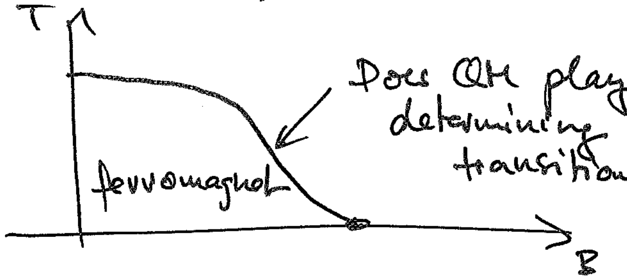
$d > d_c^+$: fluctuations are irrelevant above the upper critical dimension d_c^+ (Mean-field theory works)

$d_c^- < d < d_c^+$: phase transition exists but critical behavior is different from mean field theory (d_c^- lower critical dimension)

$d < d_c^-$: fluctuations are so strong that they completely suppress the ordered phase

1.4. How important is quantum mechanics

- QM can be necessary to explain properties of the ordered phase, e.g. superfluid is consequence of Bose-Einstein quantum statistics !
- How important is QM for the asymptotic behavior close to a thermal phase transition ?



Does QM play a ~~role~~ role in determining the nature of this transition ?

The answer is NO !

- How to distinguish fluctuations that are predominantly thermal and quantum in character ?

  $\rightarrow$ compare their thermal energy $k_B T$ with the quantum energy scale $\hbar\omega_c = \hbar/\tau_c$

  $k_B T \gg \hbar\omega_c$: thermal

  $k_B T \ll \hbar\omega_c$: quantum

- As $t \to 0_+$ the only relevant time scale is the correlation time τ_c (typical time scale for decay of fluctuations). In the ordered phase, there are long range correlations of the order parameter in time. τ_c must diverge as $t \to 0_+$:

$$\tau_c \sim \xi^z \sim |t|^{-\nu z}$$

, z dynamical critical exponent

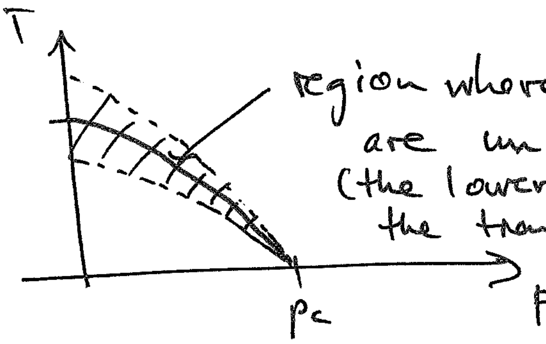
- Consider phase transition at finite $T = T_c > 0$:

  close to T_c : $\quad k_B T \simeq k_B T_c$

  but $\quad \hbar\omega_c = \hbar/\tau_c \sim |t|^{\nu z} \xrightarrow[t \to 0_+]{} 0$

  $\Rightarrow$ Sufficiently close to the transition, we will be always in the regime $k_B T \gg \hbar\omega_c$!

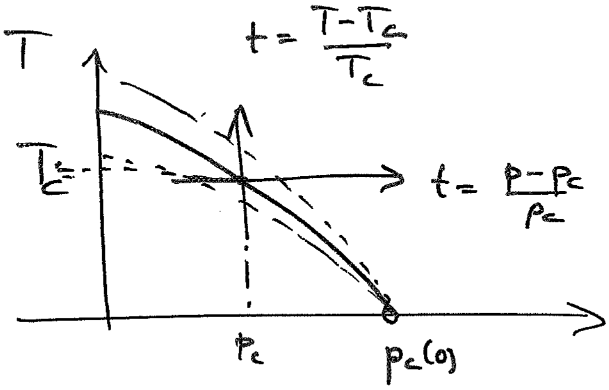
→ At thermal phase transition, quantum fluctuations play no role for the critical behavior!
(However, they influence the value of T_c)



region where quantum fluctuations are unimportant
(the lower T, the closer we have go to the transition to enter the regime $k_B T \gg \hbar\omega_c \sim |t|^{\nu z}$)

- At $T=0$, we are always in the quantum regime and the transition is driven by quantum fluctuations

- Note: It doesn't matter how we cross the phase transition:



1.5. Sketch of the quantum-to classical mapping

- Central for our understanding of quantum phase transitions and their signatures at finite temperature is the analogy between quantum systems and higher dimensional classical systems

- Will be explained in detail in next chapters, here only a sketch

Consider for example Ising model in transversal field.
On every site, Hilbert space with two states

$$|m\rangle = |1/2 \; m\rangle = \begin{cases} |\uparrow\rangle \\ |\downarrow\rangle \end{cases}$$

→ quantum state of the system:

$$|\{m_i\}\rangle = \prod_i |m_i\rangle_i$$ where i labels lattice sites of d-dim. lattice

<u>partition function</u>: $Z = \mathrm{Tr}\, e^{-\beta \hat{H}}$

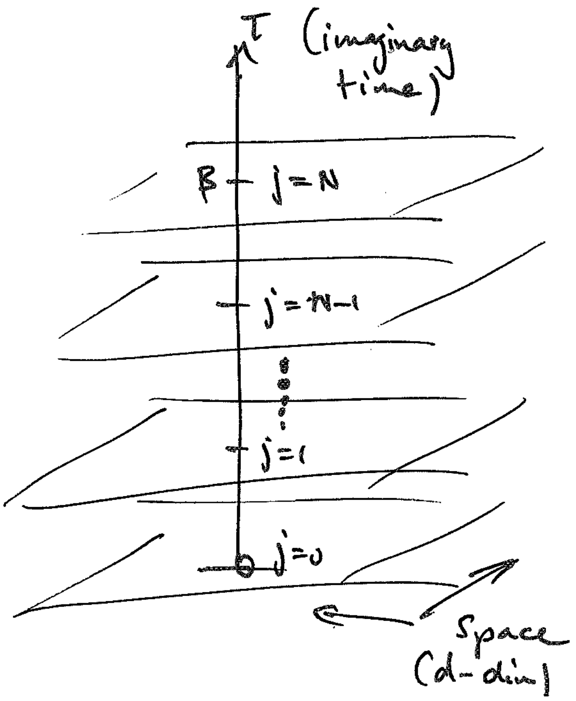
$$= \sum_{\{m_i\}} \langle \{m_i\} | \underbrace{e^{-\beta \hat{H}}}_{e^{-\beta/N \hat{H}} \cdot e^{-\beta/N \hat{H}} \cdot e^{-\beta/N \hat{H}} \cdots e^{-\beta/N \hat{H}}} | \{m_i\} \rangle$$

(N large number)

insert identities

$$1 = \sum_{\{m_i^{(1)}\}} |\{m_i^{(1)}\}\rangle\langle\{m_i^{(1)}\}|$$

$$1 = \sum_{\{m_i^{(2)}\}} |\{m_i^{(2)}\}\rangle\langle\{m_i^{(2)}\}|$$



- d-dim lattice turns into a (d+1)-dim. lattice with classical Ising variables $m_i^{(j)} = \pm 1$ on every site
- We call extra dimension <u>imaginary time</u> since $e^{-\tau \hat{H}}$ looks like unitary time evolution operator if we replace τ by it

- Thickness of imaginary-time direction is given by inverse temperature $\beta = \frac{1}{k_B T}$
- $T=0$: system becomes truly (d+1)-dimensional

- Since τ_c and ξ are the only relevant time and length scales close to the transition we can again go to an effective long wave-length description in the continuum:

$$\boxed{\begin{aligned} Z &= \int \mathcal{D}[\vec{\phi}]\, e^{-S[\vec{\phi}]} \\ S[\vec{\phi}] &= \int_0^\beta d\tau \int d^d r \left\{ |\nabla\vec{\phi}|^2 + |\partial_\tau\vec{\phi}|^2 + t|\vec{\phi}|^2 + u|\vec{\phi}|^4 - \vec{B}\vec{\phi} \right\} \end{aligned}}$$

Ginzburg – Landau functional

- At $T=0$ ($\beta=\infty$), $S[\vec{\phi}]$ is equivalent to a free-energy functional in $D=d+1$
- In the above action, time and space directions are equivalent $\Rightarrow$ $z=1$
- This is not always the case! In general, time scales as the z-th power of length and

$$D = d+z$$

$\Rightarrow$ Classical phase transition in $D=d+z$ dimensions equivalent to the quantum phase transition in d dimensions

- By analogy, the homogeneity relation for the free energy density at $T=0$ can be written as

$$f(t,B) = b^{-(d+z)} f(t b^{1/\nu}, B b^{y_B})$$

with $t = \dfrac{g-g_c}{g_c}$

- At finite T we can generalize to

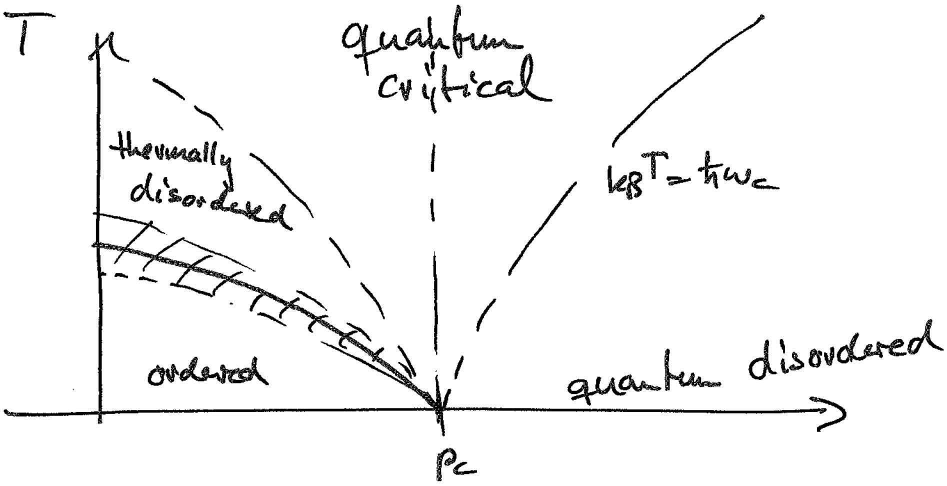
$$f(t,B) = b^{-(d+z)} f(t b^{1/\nu}, B b^{y_B}, T b^{z})$$

- In real life, we are always at finite T! Why are quantum-phase transitions more than an academic question?

a) Because there is an interesting finite-temperature crossover and quantum criticality can be observed up to amazingly high temperatures!

Compare length of imaginary time axis with time scale τ_c of critical fluctuations:

$$\tau_c = \frac{\hbar}{k_B T} \quad \Leftrightarrow \quad k_B T = \hbar\omega_c \text{ (old crossover condition)}$$



Quantum critical region:

+ length of imaginary time axis shorter than time scale of critical fluctuations

→ system looks critical

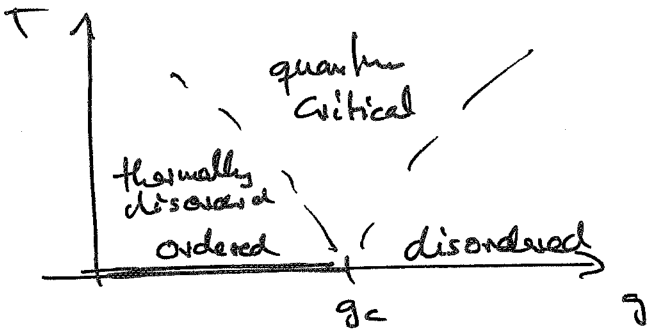
+ $k_B T \gg \hbar\omega_c = \hbar/\tau_c \sim |p-p_c|^{\nu z}$

→ dominant fluctuations are thermal but the ground state is a scale invariant quantum-critical state!

+ temperature enters as the only scale!

→ thermodynamics characterized by interesting power laws

- Note that in some cases $D = d+z$ is above the lower critical dimension d_c^- (ordered quantum phase at $T=0$) whereas $d < d_c^-$ and therefore there is no order at $T>0$



1.6. What's different about quantum-phase transitions?

Why do we need a seperate course on quantum phase transitions? Can't we just use the quantum-to-classical mapping and use the results for classical phase transitions?

The answer is NO and that's why Quantum phase transition are exciting!

I. In many cases $z \neq 1$. Therefore, time and length scale differently and the corresponding classical theory is very unusual and anisotropic. However, this is not a big problem, since we can still use the same techniques (eg. the renormalization group) as for classical phase transitions. (Example for $z=2$ in Chapter 3)

II. Sometimes the quantum-to-classical mapping leads to complex valued Boltzmann weights in the "classical theory". The imaginary parts are a consequence of the underlying quantum mechanics and called Berry-phases.

The existance of such Berry phases can lead to a whole new class of phenomena which have no analogs in classical theories. (Example will be given in Chapter 4)

III. For <u>fermions</u>, the quantum-to-classical mapping leads to negative Boltzmann weights (fermion sign problem). This is the reason why the understanding of strongly correlated electron systems (e.g. the high-T_c superconductors) is still one of the big open problems of contemporary physics.