2. Conservation Laws

Symmetries $\Longleftrightarrow$ conservation laws (Noether's theorem)

2.1. Energy

- Time invariance: $L = L(q, \dot{q})$ (no explicit dependence on time)

$$\frac{dL}{dt} = \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \underset{\text{E.L.}}{=} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q}$$

$$= \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}}\right)$$

$$\Longrightarrow \frac{d}{dt}\left(\dot{q}\frac{\partial L}{\partial \dot{q}} - L\right) = 0 \Longrightarrow \dot{q}\frac{\partial L}{\partial \dot{q}} - L = \sum_{\alpha=1}^{N} \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L = \text{const}$$

We have found a constant of motion. Note that we have used this to solve the Brachistochrone problem. What is the meaning of this constant?

- Evaluate for interacting mass points,

$$L = \sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2 - U(\underline{r}_1, \dots, \underline{r}_n)$$

$$\dot{q}\frac{\partial L}{\partial \dot{q}} - L = \sum_i m_i \dot{\underline{r}}_i^2 - \left(\sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2 - U(\underline{r}_1, \dots, \underline{r}_n)\right)$$

$$= \underbrace{\sum_i \frac{m_i}{2}\dot{\underline{r}}_i^2}_{\text{kinetic energy}} + \underbrace{U(\underline{r}_1, \dots, \underline{r}_n)}_{\text{potential energy}} = \text{total energy}$$

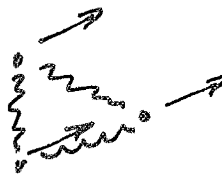
- Time invariance $\longleftrightarrow$ energy conservation

$$\boxed{E = \dot{q}\frac{\partial L}{\partial \dot{q}} - L = \sum_\alpha \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L\,, \quad \frac{dE}{dt} = 0}$$

2.2. Momentum

- Homogeneous Space:
  It doesn't matter where you put a system of interacting particles.
  $\rightarrow$ Invariance under spatial translations

$$\forall_{i=1,\dots,n}: \quad \underline{r}_i' = \underline{r}_i + \underbrace{\delta\underline{r}}_{\text{small and constant}}$$



Invariance: Equations of motion unchanged

$$\begin{aligned} L' &= L(\underline{r}_i + \delta\underline{r}, \dot{\underline{r}}_i, t) \\ &= L + \sum_i \frac{\partial L}{\partial \underline{r}_i}\delta\underline{r} + \mathcal{O}(\delta\underline{r}^2) \end{aligned}$$

$$\Rightarrow \quad \delta L = \delta\underline{r} \cdot \sum_i \frac{\partial L}{\partial \underline{r}_i} \overset{!}{=} 0$$

This should equal zero for all $\delta\underline{r}$!

$$\Rightarrow 0 = \sum_i \frac{\partial L}{\partial \underline{r}_i} \underset{\text{E.L.}}{=} \frac{d}{dt}\underbrace{\sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i}}_{\text{constant of motion}}$$

$$\boxed{\underline{P} = \sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i} = \sum_i \underline{p}_i\,, \quad \frac{d\underline{P}}{dt} = 0}$$ momentum conservation

- interacting mass points, $L = \sum_i \frac{m_i}{2} \dot{\underline{r}}_i^2 - U(\underline{r}_1, \dots, \underline{r}_n)$

$$\underline{P} = \sum_i \underline{p}_i \quad , \quad \underline{p}_i = \frac{\partial L}{\partial \dot{\underline{r}}_i} = m_i \dot{\underline{r}}_i$$

We can also derive Newton's 3rd law („actio = reactio"):

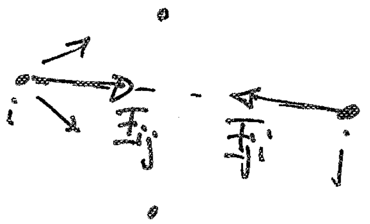
$$0 \stackrel{!}{=} U(\underline{r}_1 + \delta\underline{r}, \dots, \underline{r}_n + \delta\underline{r}) - U(\underline{r}_1, \dots, \underline{r}_n)$$

$$= \sum_i \frac{\partial U}{\partial \underline{r}_i} \cdot \delta\underline{r}$$

$\underset{\delta\underline{r} \text{ arbitrary}}{\Longrightarrow}$ $\quad 0 = \sum_i \frac{\partial U}{\partial \underline{r}_i} \Rightarrow \sum_i \underline{F}_i = 0$

$\underline{F}_i = -\underline{\nabla}_i U$ $\quad$ no net force!

(only internal forces)

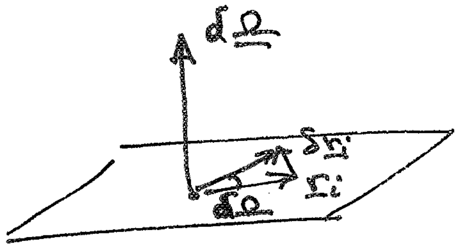


$$\underline{F}_i = \sum_{j(\neq i)} \underline{F}_{ij}$$

$$\sum_i \underline{F}_i = 0 \Rightarrow \boxed{\underline{F}_{ij} = -\underline{F}_{ji}}$$

2.3. Angular Momentum

Isotropic Space: Invariance with respect to arbitrary infinitesimal rotations



$d\Omega = |d\underline{\Omega}|$ rotation angle

$\delta\underline{\Omega}$ orthogonal to rotation plane

$$\boxed{\delta\underline{r}_i = \delta\underline{\Omega} \times \underline{r}_i}$$

$$L' = L(\underline{r}_i + \delta\underline{r}_i, \dot{\underline{r}}_i + \delta\dot{\underline{r}}_i, t) = L + \sum_i \left(\frac{\partial L}{\partial \dot{\underline{r}}_i} \delta\dot{\underline{r}}_i + \frac{\partial L}{\partial \underline{r}_i} \delta\underline{r}_i \right)$$

$$\Rightarrow \quad \delta L = \sum_i \left(\frac{\partial L}{\partial \dot{\underline{r}}_i} \delta\dot{\underline{r}}_i + \frac{\partial L}{\partial \underline{r}_i} \delta\underline{r}_i \right)$$

$$= \sum_i \left(\frac{\partial L}{\partial \dot{\underline{r}}_i} (\delta\underline{\Omega} \times \dot{\underline{r}}_i) + \frac{\partial L}{\partial \underline{r}_i} (\delta\underline{\Omega} \times \underline{r}_i) \right)$$

$$\underset{\substack{\underline{a}(\underline{b}\times\underline{c}) \\ = \underline{c}(\underline{a}\times\underline{b}) \\ = \underline{b}(\underline{c}\times\underline{a})}}{=} \delta\underline{\Omega} \cdot \sum_i \left(\dot{\underline{r}}_i \times \frac{\partial L}{\partial \dot{\underline{r}}_i} + \underline{r}_i \times \frac{\partial L}{\partial \underline{r}_i} \right) \overset{!}{=} 0$$

$$\overset{E.L.}{=} \delta\underline{\Omega} \sum_i \left(\dot{\underline{r}}_i \times \frac{\partial L}{\partial \dot{\underline{r}}_i} + \underline{r}_i \times \frac{d}{dt} \frac{\partial L}{\partial \dot{\underline{r}}_i} \right)$$

$$= \delta\underline{\Omega} \cdot \frac{d}{dt} \sum_i \underline{r}_i \times \underbrace{\frac{\partial L}{\partial \dot{\underline{r}}_i}}_{\underline{p}_i} \overset{!}{=} 0$$

$\underset{\delta\underline{\Omega}\text{ arbitrary}}{\Rightarrow}$ conservation of angular momentum

$$\boxed{\underline{L} = \sum_i \underline{L}_i = \sum_i \underline{r}_i \times \underline{p}_i, \quad \frac{d\underline{L}}{dt} = 0}$$

2.4. Center of Mass

Center of mass: $\underline{R} = \frac{1}{M} \sum_i m_i \underline{r}_i$

total mass: $M = \sum_i m_i$

total momentum: $\underline{P} = \sum_i m_i \dot{\underline{r}}_i$

$$\underline{R}_0 = \underline{R}(t) - \frac{\underline{P}}{M} t$$

$$\frac{d\underline{R}_0}{dt} = \frac{1}{M}\left(\sum_i m_i \dot{\underline{r}}_i - \underline{P}\right) = 0$$

free motion of center of mass

$\Leftrightarrow$ conservation of initial center of mass

- It turns out that this conservation law is related to the Galilean invariance

2.5. Symmetries and Conservation Laws

Symmetry	Conservation Law	Components
time	energy $E = \sum_i \dot{\underline{r}}_i \frac{\partial L}{\partial \dot{\underline{r}}_i} - L$	1
homogeneous space (spatial translations)	momentum $\underline{P} = \sum_i \underline{p}_i = \sum_i \frac{\partial L}{\partial \dot{\underline{r}}_i}$	3
isotropic space (rotations)	angular momentum $\underline{L} = \sum_i \underline{L}_i = \sum_i \underline{r}_i \times \underline{p}_i = \sum_i \underline{r}_i \times \frac{\partial L}{\partial \dot{\underline{r}}_i}$	3
Galilean Invariance	center of mass $\underline{R}_0 = \underline{R} - \frac{\underline{P}}{M} t$, $\underline{R} = \frac{1}{M}\sum_i m_i \underline{r}_i$	3
		$\sum_i = 10$

10 fundamental mechanical quantities

10 " spacetime symmetries