Chapter 2

Multidimensional Integration

2.1 Line Integrals

2.1.1 Mathematical Concept

If we have a vector field $\mathbf{G}(\mathbf{r})$, then we can define a *line integral*

$$I = \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r} \tag{2.1}$$

along any path (or contour) C going from an initial point $\mathbf{r}_A$ to a final point $\mathbf{r}_B$. The mathematical definition of a line integral is obtained by breaking the path into small displacement elements $d\mathbf{r}$, and defining the line integral to be the sum of all the elementary contributions $\mathbf{G} \cdot d\mathbf{r}$ in the limit where division of the path into elements becomes infinitely fine. In this limit, the lengths of all the elements go to zero and the sum becomes and integral.

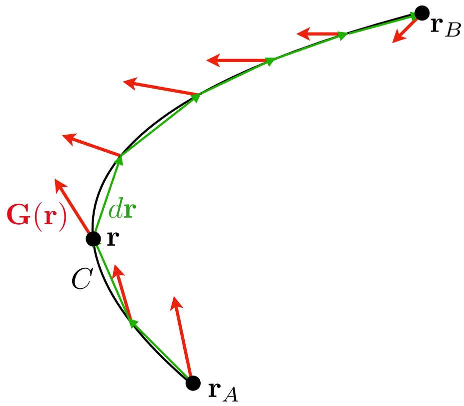


Figure 2.1: Illustration of the line integral $I = \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r}$.

Note two important things:

- The result of a line integral is a scalar, not a vector; this is because the line integral contains a scalar product $\mathbf{G} \cdot d\mathbf{r}$ of the vector filed $\mathbf{G}$ and the infinitesimal displacement vector $d\mathbf{r}$.
- The notation for a line integral includes a specification of the path C, which is written as a subscript on the integral sign. In general, a line integral depends on the initial and final points $\mathbf{r}_A$ and $\mathbf{r}_B$ *and* on the choice of path running from $\mathbf{r}_A$ to $\mathbf{r}_B$.

A special class of line integrals are so called *loop integrals*, for which the initial and final points are the same: $\mathbf{r}_A = \mathbf{r}_B$. The path running from $\mathbf{r}_A$ to $\mathbf{r}_B$ then forms a closed loop. A special notation is sometimes used for a loop integral, consisting of a normal integral sign with a circle drawn in the middle of it,

$$I = \oint_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r} \ .$$

2.1.2 Work Done in Moving a Particle Along a Path

Consider a particle in a force field $\mathbf{F}(\mathbf{r})$. If the particle is moved by an infinitesimal displacement vector $d\mathbf{r}$ from $\mathbf{r}$ to $\mathbf{r} + d\mathbf{r}$, the *work* done is equal to

$$dW = -\mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}. \tag{2.2}$$

Now suppose that we repeat this process a large number of times to move the particle along a given path C from an initial position $\mathbf{r}_A$ to a final position $\mathbf{r}_B$. The total work done is then given by the sum of all the small contributions dW. As stated above, this the definition of a *line integral*,

$$W^C_{A\to B} = \int_C dW = -\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}. \tag{2.3}$$

2.1.3 Practical Evaluation of Line Integrals

If you have a vector field $\mathbf{G}$, which is a given function of x and y (and perhaps z), how do you evaluate its line integral along a given path C?

The key step is to *choose a parameter specifying the position on the path*, and express the line integral as an ordinary integral with respect to this parameter. Let us assume that we found such a *parametrisation*,

$$\begin{array}{rcl} t & \longrightarrow & \mathbf{r}(t) \\ {[a,b]} & \longrightarrow & \mathbb{R}^3 \end{array}$$

of the path C such that $\mathbf{r}(a) = \mathbf{r}_A$ and $\mathbf{r}(b) = \mathbf{r}_B$. We will give examples of various parametrisations of straight and curved paths in two and three dimensions later. Since the positions along the path C are specified by a single parameter t, we can express the vector field for each point on the path as a function of a single variable t, $\mathbf{G} = \mathbf{G}(\mathbf{r}(t))$. The line element $d\mathbf{r}$ along C can evaluated as

$$d\mathbf{r}(t) = \begin{pmatrix} dx(t) \\ dy(t) \\ dz(t) \end{pmatrix} = \begin{pmatrix} x'(t)dt \\ y'(t)dt \\ z'(t)dt \end{pmatrix} = \mathbf{r}'(t)\,dt.$$

We can therefore express the line integral as an ordinary, one dimensional integral over the parameter t,

$$I = \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt, \tag{2.4}$$

where $\mathbf{r}(t)$ is a parametrisation of the path C from $\mathbf{r}(a) = \mathbf{r}_A$ to $\mathbf{r}(b) = \mathbf{r}_B$.

Example: We want to evaluate the line integral

$$I = \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r},$$

with

$$\mathbf{G}(\mathbf{r}) = \mathbf{G}(x, y) = \begin{pmatrix} xy \\ -y^2 \end{pmatrix} = xy\,\hat{\mathbf{e}}_x - y^2\,\hat{\mathbf{e}}_y \tag{2.5}$$

for two different paths:

(a) the straight-line from the initial point $\mathbf{r}_A = (0, 0)$ to the final point $\mathbf{r}_B = (2, 1)$;

(b) the parabolic path $y = x^2/4$ joining the same two points.

The crucial step is to choose a parameter to specify the position on the path. If the path is determined by a function $y = f(x)$ for $x \in [a, b]$, it is convenient to use the x coordinate. In this case

$$\mathbf{r}(x) = \begin{pmatrix} x \\ f(x) \end{pmatrix}.$$

(a) The straight line from $(0, 0)$ to $(2, 1)$ is given by the function $y = \frac{1}{2}x$ for $x \in [0, 2]$. Using the parametrisation

$$\mathbf{r}(x) = \begin{pmatrix} x \\ \frac{1}{2}x \end{pmatrix},$$

we obtain

$$\begin{aligned} I &= \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^2 \mathbf{G}(\mathbf{r}(x)) \cdot \mathbf{r}'(x)dx = \int_0^2 \begin{pmatrix} \frac{1}{2}x^2 \\ -\frac{1}{4}x^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} dx \\ &= \int_0^2 \left(\frac{1}{2}x^2 - \frac{1}{8}x^2\right) dx = \int_0^2 \frac{3}{8}x^2 dx = \left[\frac{1}{8}x^3\right]_0^2 = 1. \end{aligned}$$

Important note: The value of a line integral along a path C is *independent* of the choice of parametrisation of the path, as it is implied by the general definition (2.1.1). There exist in fact many different parametrisations of the same path. E.g., instead of the parametrisation used above we can parametrise the straight line from $(0,0)$ to $(2,1)$ by

$$\mathbf{r}(t) = t\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}2t\\t\end{pmatrix},$$

with $t \in [0,1]$. Using this parametrisation we obtain

$$I = \int_0^1 \mathbf{G}(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_0^1 \begin{pmatrix}2t^2\\-t^2\end{pmatrix}\cdot\begin{pmatrix}2\\1\end{pmatrix}dt = \int_0^1 3t^2dt = \left[t^3\right]_0^1 = 1,$$

in agreement with the previous calculation.

(b) For $y = \frac{1}{4}x^2$, $x \in [0,2]$ we have

$$\mathbf{r}(x) = \begin{pmatrix}x\\\frac{1}{4}x^2\end{pmatrix}, \quad \mathbf{G}(\mathbf{r}(x)) = \begin{pmatrix}\frac{1}{4}x^3\\-\frac{1}{16}x^4\end{pmatrix},$$

and hence

$$\begin{aligned}I &= \int_C \mathbf{G}(\mathbf{r})\cdot d\mathbf{r} = \int_0^2 \mathbf{G}(\mathbf{r}(x))\cdot\mathbf{r}'(x)dx = \int_0^2 \begin{pmatrix}\frac{1}{4}x^3\\-\frac{1}{16}x^4\end{pmatrix}\cdot\begin{pmatrix}1\\\frac{1}{2}x\end{pmatrix}dx\\ &= \int_0^2\left(\frac{1}{4}x^3 - \frac{1}{32}x^5\right) = \left[\frac{1}{16}x^4 - \frac{1}{192}x^6\right]_0^2 = 1 - \frac{64}{192} = 1 - \frac{1}{3} = \frac{2}{3}.\end{aligned}$$

Note that even though the two paths that we have just considered run from the same initial point $(0,0)$ to the same final point $(2,1)$, the numerical values of the line integrals are not the same.

2.1.4 Conservative Vector Fields

There is an important class of vector fields, called "conservative vector fields", for which line integrals do not depend on which path is taken from $\mathbf{r}_A$ to $\mathbf{r}_B$, and for which all loop integrals are zero.

The vector field $\mathbf{G}(\mathbf{r}) = xy\,\hat{\mathbf{e}}_x - y^2\,\hat{\mathbf{e}}_y$, Eq. (2.5), is not conservative since the line integrals along two different paths from $(0,0)$ to $(2,1)$ gave different values. How to test wether a given vector field is conservative? Of course it is not possible show that for *all* paths from $\mathbf{r}_A$ to $\mathbf{r}_B$ the line integral gives the same value, simply because there exist infinitely many possible paths. Instead we can use an equivalent definition of conservative fields:

$$\text{The vector field } \mathbf{G}(\mathbf{r}) \text{ is conservative} \iff \exists \text{ scalar field } \phi(\mathbf{r}) : \mathbf{G} = \nabla\phi \qquad (2.6)$$

The field $\phi(\mathbf{r})$ is called a *potential* of $\mathbf{G}(\mathbf{r})$.[1] Let us prove that for such vector fields line integrals are indeed independent of the path taken from $\mathbf{r}_A$ to $\mathbf{r}_B$. Consider a path C parametrised by $\mathbf{r}(t)$ with $t \in [a,b]$ and $\mathbf{r}(a) = \mathbf{r}_A$ and $\mathbf{r}(b) = \mathbf{r}_B$. The line integral along the path C is given by

$$\begin{aligned} I \quad &= \quad \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r} = \int_a^b \mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_a^b \nabla\phi(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &\overset{d\phi = \nabla\phi\cdot d\mathbf{r}}{=} \quad \int_a^b \frac{d}{dt}\phi(\mathbf{r}(t))\, dt = [\phi(\mathbf{r}(t))]_a^b = \phi(\mathbf{r}_B) - \phi(\mathbf{r}_A). \end{aligned}$$

This result shows that the line integral is equal to the difference of the potentials at the final and initial positions and therefore independent of the path connecting these points. It is obvious that for any closed loop ($\mathbf{r}_A = \mathbf{r}_B$) the line integral along the loop vanishes. For a three dimensional vector field defined on a domain that is 'simply connected' (contains no holes) there exist a third, equivalent definition of conservativeness:

$$\text{The vector field } \mathbf{G}(\mathbf{r}) \text{ is conservative} \iff \text{curl}\, \mathbf{G} = \nabla \times \mathbf{G} = 0. \qquad (2.7)$$

Example: Let us test if the vector field $\mathbf{G}(\mathbf{r}) = 2xy\, \hat{\mathbf{e}}_x + (x^2 - 2y)\, \hat{\mathbf{e}}_y + \hat{\mathbf{e}}_z$ is conservative. Since $\mathbf{G}$ is defined on the entire three-dimensional space $\mathbb{R}^3$ we can use the "curl" criterion,

$$\nabla \times \mathbf{G} = \begin{pmatrix} \partial_y G_z - \partial_z G_y \\ \partial_z G_x - \partial_x G_z \\ \partial_x G_y - \partial_y G_x \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ 0 - 0 \\ 2x - 2x \end{pmatrix} = \mathbf{0}.$$

Since the curl of $\mathbf{G}$ is zero, $\mathbf{G}$ is a conservative vector field. It must therefore be possible to find a potential $\phi(\mathbf{r})$ such that $\mathbf{G} = \nabla\phi$. Let us construct such a scalar field $\phi(\mathbf{r})$. We need to find a function of x, y and z that satisfies the three equations

$$\text{I. } \partial_x\phi = 2xy, \quad \text{II. } \partial_y\phi = x^2 - 2y, \quad \text{III. } \partial_z\phi = 1.$$

From the first equation we obtain by integration that $\phi(\mathbf{r}) = x^2y + f(y,z)$. Here $f(y,z)$ can be any function of y and z since it is treated as constant when taking a partial derivative with respect to x. Integrating all three equations we obtain

$$\text{I. } \phi = x^2y + f(y,z), \quad \text{II. } \phi = x^2y - y^2 + g(x,z), \quad \text{III. } \phi = z + h(x,y).$$

Any potential satisfying all three equations must be of the form

$$\phi(\mathbf{r}) = x^2y - y^2 + z + c,$$

with $c \in \mathbb{R}$ an additive constant. Note that the constant c drops out when calculating a line integral (difference of potentials of final and initial points).

[1] Note that the *potential energy* $U(\mathbf{r})$ of a conservative force field $\mathbf{F}(\mathbf{r})$ is defined with an extra minus sign, $\mathbf{F} = -\nabla U$. This minus sign compensates the minus sign in the definition of the line integral for the work done in the force field. The resulting total work is given by the difference of potential energies, $W_{A\to B} = U(\mathbf{r}_B) - U(\mathbf{r}_A)$.

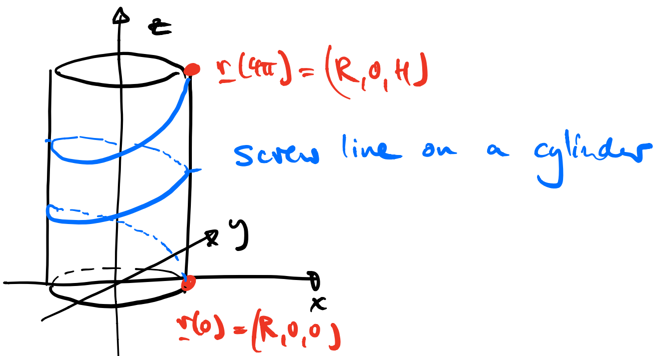
2.1.5 Worked Example

Sketch the contour C that is parametrised by

$$\mathbf{r}(t) = \begin{pmatrix} R\cos(t) \\ R\sin(t) \\ \frac{H}{4\pi}t \end{pmatrix}$$

for $t \in [0, 4\pi]$. Calculate the line integral $I = \int_C \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r}$ along this contour, where the vector field is given by

$$\mathbf{G}(\mathbf{r}) = \begin{pmatrix} y \\ -x \\ z \end{pmatrix}.$$



$$\mathbf{G}(\mathbf{r}(t)) = \begin{pmatrix} R\sin(t) \\ -R\cos(t) \\ \frac{H}{4\pi}t \end{pmatrix}; \quad \mathbf{r}'(t) = \begin{pmatrix} -R\sin(t) \\ R\cos(t) \\ H/4\pi \end{pmatrix}$$

$$I = \int_0^{4\pi} \mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_0^{4\pi} \begin{pmatrix} R\sin(t) \\ -R\cos(t) \\ \frac{H}{4\pi}t \end{pmatrix} \cdot \begin{pmatrix} -R\sin(t) \\ R\cos(t) \\ H/4\pi \end{pmatrix} dt$$

$$= \int_0^{4\pi} \left[\left(\frac{H}{4\pi}\right)^2 t - R^2 \right] dt = \left[\frac{1}{2}\left(\frac{H}{4\pi}\right)^2 t^2 - R^2 t \right]_0^{4\pi}$$

$$= \frac{1}{2}H^2 - 4\pi R^2$$

2.2 Area Integrals

2.2.1 Mathematical Concept

For any scalar field $f(\mathbf{r})$ of the 2-dimensional position $\mathbf{r}$, we can define an area integral

$$I = \int_A f(\mathbf{r})\, dA, \qquad (2.8)$$

where the integral goes over a specified 2-dimensional region A. It is defined conceptually in a similar way to the line integral: the region is divided up into *area elements* dA, in each of which the function has some value $f(\mathbf{r})$. We add up all the elementary contributions $f(\mathbf{r})\, dA$ and take the limit that the area elements become infinitesimally small.

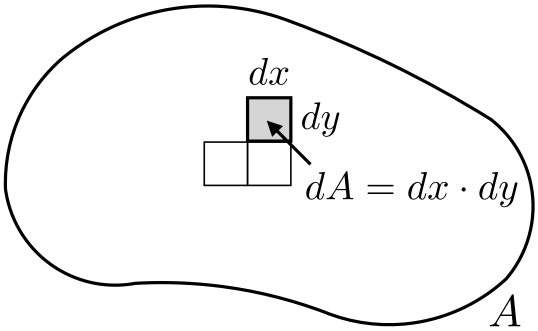


Figure 2.2: Division of a 2-dimensional area A into infinitesimal area elements dA.

An area integral is a *double* integral, because we have to perform integrals over both the x and y variables. When integrating over one variable, the other is held constant.

Note that an important special case is $f(\mathbf{r}) \equiv 1$. In this case, the area integral is equal to the total area, $A = \int_A dA$.

2.2.2 Area Integrals in Physics

As an example of an area integral in physics we consider a flat plate with electric charge spread on its surface. To start with, take a rectangular plate in the x-y plane. The corners of the rectangle are at the points $(0, 0)$, $(a, 0)$, (a, b) and $(0, b)$, where a and b are the lengths of the sides of the rectangle. The charge density (charge per unit area) is called $\sigma(\mathbf{r}) = \sigma(x, y)$.
We'd like to calculate the total charge Q on the plate, given by the area integral

$$Q = \int_A \sigma(\mathbf{r}) dA = \int_{x=0}^{x=a} \int_{y=0}^{y=b} \sigma(x, y)\, dxdy = \int_0^a dx \int_0^b dy\, \sigma(x, y). \qquad (2.9)$$

Note that the latter notation for the double integral is commonly used for brevity.

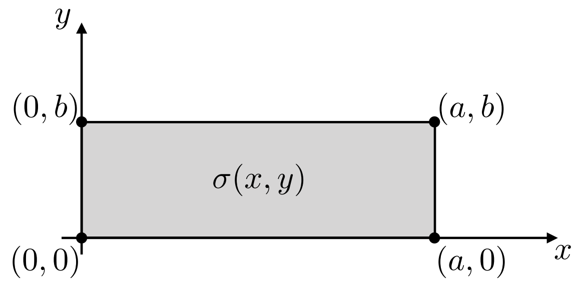


Figure 2.3: Charge density $\sigma(x, y)$ on a rectangular plate.

Example 1: charge is spread evenly, so that charge density is uniform everywhere on the plate. This means that σ is a constant (no dependence on x and y). We obtain

$$Q = \sigma \int_0^a dx \int_0^b dy = \sigma ab = \sigma A.$$

Of course, in this simple case the homogeneous charge density is equal to the total charge divided by the total area, $\sigma = Q/A$.

Example 2: the charge density is non-uniform and given by the function $\sigma(x, y) = xy + y^2$. We could first calculate the y-integral, keeping x constant,

$$Q = \int_0^a dx \int_0^b dy \left(xy + y^2\right) = \int_0^a dx \left[\frac{1}{2}xy^2 + \frac{1}{3}y^3\right]_{y=0}^{y=b} = \int_0^a dx \left(\frac{1}{2}b^2 x + \frac{1}{3}b^3\right).$$

This one-dimensional integral has a simple interpretation. It is equal to the sum of the charges on strips of infinitesimal thickness dx, running parallel to the y-axis. After performing the remaining x integral we obtain

$$Q = \left[\frac{1}{4}b^2x^2 + \frac{1}{3}b^3 x\right]_{x=0}^{x=a} = \frac{1}{4}a^2b^2 + \frac{1}{3}ab^3.$$

Note that we could also perform the x and y integrals in the opposite order, starting with the x integral,

$$Q = \int_0^b dy \int_0^a dx \left(xy + y^2\right) = \int_0^b dy \left[\frac{1}{2}x^2 y + xy^2\right]_{x=0}^{x=a} = \int_0^b dy \left(\frac{1}{2}a^2 y + ay^2\right).$$

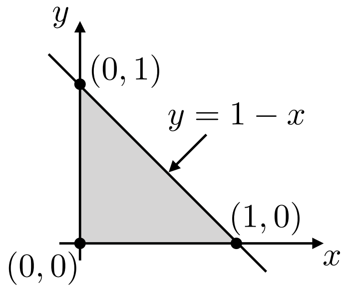
The remaining integral corresponds to a sum of charges on strips of infinitesimal thickness dy, running parallel to the x-axis. Performing the y integral, we obtain the same final result as before,

$$Q = \left[\frac{1}{4}a^2y^2 + \frac{1}{3}ay^3\right]_{y=0}^{y=b} = \frac{1}{4}a^2b^2 + \frac{1}{3}ab^3.$$

As one might have expected, it does not matter in which order the contributions from the area elements are added. And, as we shall see a bit later, the value of an area integral does not depend on the coordinates we use to describe $\mathbf{r}$ or dA. This is both implied by the general definition $I = \int f(\mathbf{r})dA$ of the area integral.

2.2.3 Non-Rectangular Regions

The region A in the plane over which we integrate does not have to be a rectangle. Let's for example integrate the function $f(x, y) = x^2 + 2xy$ over the interior of a triangle whose corners are at the points $(0, 0)$, $(1, 0)$, and $(0, 1)$.



To do this, we can imagine that we divide the triangular region into strips running parallel to the y-axis. Because the region is a triangle, the strips have different lengths: the strip at position $x \in [0, 1]$ runs from $y = 0$ to $y = 1 - x$. So we can write the area integral as the double integral

$$I = \int_0^1 dx \int_0^{1-x} dy \left(x^2 + 2xy\right) . \tag{2.10}$$

This says: integrate first over y with x held constant, the limits on y being $[0, 1-x]$. Then integrate over x, with limits $[0, 1]$. Let's perform these integrations:

$$\begin{aligned} I &= \int_0^1 dx \left[x^2 y + xy^2\right]_{y=0}^{y=1-x} \\ &= \int_0^1 dx \, \left[x^2(1-x) + x(1-x)^2\right] \\ &= \int_0^1 dx \, (x - x^2) \\ &= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{x=0}^{x=1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}. \end{aligned}$$

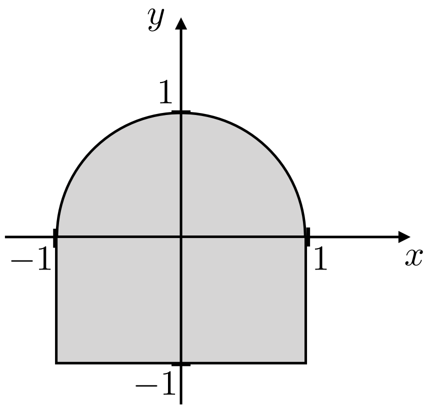
Using that $y = 1 - x \Leftrightarrow x = 1 - y$ we can also perform the integrations in the opposite order,

$$\begin{aligned} I &= \int_0^1 dy \int_0^{1-y} dx \, (x^2 + 2xy) = \int_0^1 dy \left[\frac{1}{3}x^3 + x^2 y\right]_{x=0}^{x=1-y} \\ &= \int_0^1 dy \left[\frac{1}{3}(1-y)^3 + (1-y)^2 y\right] = \int_0^1 dy \left(\frac{2}{3}y^3 - y^2 + \frac{1}{3}\right) \\ &= \left[\frac{1}{6}y^4 - \frac{1}{3}y^3 + \frac{1}{3}y\right]_{y=0}^{y=1} = \frac{1}{6} - \frac{1}{3} + \frac{1}{3} = \frac{1}{6}, \end{aligned}$$

giving the same result, as expected. By generalising this procedure, it is possible to perform an area integral over a region in the x-y plane that has a completely arbitrary shape. We just need to figure out the correct integration limits.

2.2.4 Worked Example

Perform the area integral $I = \int_A x^2 y \, dA$ over an area A that is given by the shaded region compose of a rectangle and a half circle.



For $x \in [-1,1]$, y has to be integrated from -1 to $\sqrt{1-x^2}$

$$I = \int_{-1}^{1} dx \int_{-1}^{\sqrt{1-x^2}} dy \; x^2 y$$

$$= \frac{1}{2} \int_{-1}^{1} dx \left[x^2 y^2 \right]_{-1}^{y=\sqrt{1-x^2}}$$

$$= \frac{1}{2} \int_{-1}^{1} dx \left[x^2 (1-x^2) - x^2 \right]$$

$$= -\frac{1}{2} \int_{-1}^{1} dx \; x^4 = -\frac{1}{10} \left[x^5 \right]_{-1}^{1} = -\frac{1}{5}$$

2.2.5 Polar Coordinates

In many applied problems it is often more convenient to use other *coordinate systems* that better reflect the symmetries of the problem. While the results of line, area, or volume integrals do not depend on the choice of the coordinate system, the integrations can become much simpler using more appropriate coordinates.

So far, we have used *rectangular* or *cartesian* coordinates x and y. In theses coordinates, the line and area elements are given by $d\mathbf{r} = \hat{\mathbf{e}}_x\,dx + \hat{\mathbf{e}}_y\,dy$ and $dA = dx\,dy$, respectively. In many cases we use *polar coordinates* to specify the position of every point in the 2-dimensional x-y plane. The position of the point $\mathbf{r}$ is uniquely defined by the distance r of the point to the origin (the length of the vector $\mathbf{r}$) and the angle the vector $\mathbf{r}$ encloses with the x axis.

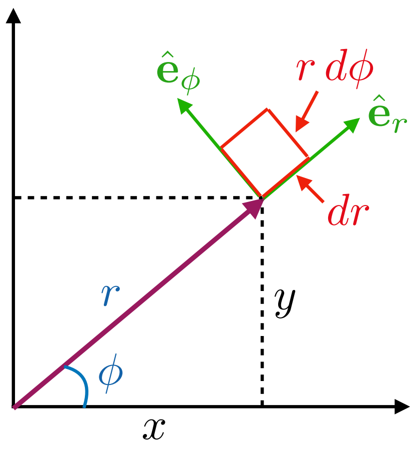


Figure 2.4: Illustration of polar coordinates.

Since x, y, and r are the sides of a right-angled triangle, we obtain $\cos\phi = x/r$ and $\sin\phi = y/r$. The cartesian coordinates (x, y) can therefore be expressed by the polar coordinates (r, ϕ) as

$$x = r\cos\phi, \quad y = r\sin\phi. \tag{2.11}$$

This is an example of a *coordinate transformation*. The *line element* in polar coordinates is given by

$$d\mathbf{r} = \hat{\mathbf{e}}_r dr + \hat{\mathbf{e}}_\phi r d\phi, \tag{2.12}$$

where we have defined the unit vectors

$$\hat{\mathbf{e}}_r = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}, \quad \hat{\mathbf{e}}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix}. \tag{2.13}$$

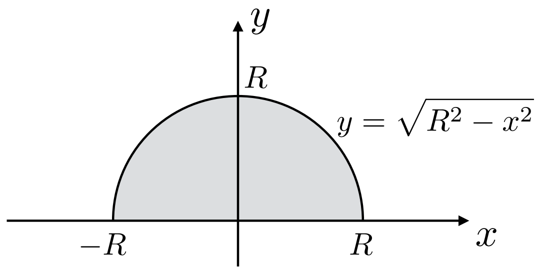
The above expression for the line element can be easily derived using the transformation (2.11),

$$\begin{aligned} d\mathbf{r} &= \begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} dr\cos\phi - r\sin\phi\, d\phi \\ dr\sin\phi + r\cos\phi\, d\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} dr + \begin{pmatrix} -\sin\phi \\ \cos\phi \end{pmatrix} rd\phi \\ &= \hat{\mathbf{e}}_r dr + \hat{\mathbf{e}}_\phi r d\phi. \end{aligned} \tag{2.14}$$

It is easy to check that the vectors $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\phi$ have length one, $|\hat{\mathbf{e}}_r| = |\hat{\mathbf{e}}_\phi| = 1$, and that $\hat{\mathbf{e}}_r \cdot \hat{\mathbf{e}}_\phi = 0$, implying that the two vectors are orthogonal. $\hat{\mathbf{e}}_r$ points along the direction of $\mathbf{r}$. To find the area element we change both polar coordinates by an infinitesimal amount dr and $d\phi$, respectively. Changing r by dr, the vector $\mathbf{r}$ changes by dr along the direction of the unit vector $\hat{\mathbf{e}}_r$. Changing ϕ by $d\phi$, the vector $\mathbf{r}$ changes by $rd\phi$ along the direction of the unit vector $\hat{\mathbf{e}}_\phi$. As illustrated in Fig. 2.4, this defines an infinitesimally small rectangular area element with side lengths dr and $rd\phi$. The area element in polar coordinates is therefore given by

$$dA = dr \cdot rd\phi = rdrd\phi. \tag{2.15}$$

To illustrate the advantage of polar coordinates in certain cases, let us calculate the area of a half circle of radius R, using both cartesian and polar coordinates.



Let us start with cartesian coordinates,

$$A = \int dA = \int_{-R}^{R} dx \int_0^{\sqrt{R^2-x^2}} dy = \int_{-R}^{R} dx \sqrt{R^2 - x^2}.$$

This integral is already not so nice. We can make a substitution, $x = R\sin t$. Using that $dx = R\cos t\, dt$, we obtain

$$A = \int_{-\pi/2}^{\pi/2} dt\, R\cos t \sqrt{R^2(1-\sin^2 t)} = R^2 \int_{-\pi/2}^{\pi/2} dt\, \cos^2 t.$$

To determine the integral of $\cos^2 t$, we make use of the double-angle relation,

$$\int \cos^2 t\, dt = \int \left(\frac{1}{2} + \frac{1}{2}\cos(2t)\right) dt = \frac{1}{2}t + \frac{1}{4}\sin(2t) = \frac{1}{2}t + \frac{1}{2}\cos t \sin t.$$

We therefore obtain

$$A = R^2 \left[\frac{1}{2} t + \frac{1}{2} \cos t \sin t \right]_{t=-\pi/2}^{t=\pi/2} = \frac{1}{2} \pi R^2.$$

In polar coordinates, the are integral becomes trivial,

$$A = \int dA = \int_0^{\pi} d\phi \int_0^R r \, dr = \pi \left[\frac{1}{2} r^2 \right]_{r=0}^{r=R} = \frac{1}{2} \pi R^2.$$

Note: the most important things when choosing the easiest coordinate system for an integral are the **limits of integration**, not the function being integrated.

2.2.6 Worked Examples

By using polar coordinates, calculate the value of the area integral

$$I = \int_A f(\mathbf{r})\, dA,$$

over the area A given by a circle of radius 2 centred at the origin for the functions

$$\text{(a) } f(\mathbf{r}) = x^2, \quad \text{(b) } f(\mathbf{r}) = \frac{x^2}{x^2+y^2}, \quad \text{and} \quad \text{(c) } f(\mathbf{r}) = r.$$

(2) $x = r\cos\phi$ Integration range: $\phi \in [0, 2\pi]$

$y = r\sin\phi$ $r \in [0, 2]$

$dA = r\,dr\,d\phi$

(a) $f(\underline{r}) = x^2 = r^2\cos^2\phi$

$$I = \int_A f(\underline{r})\, dA = \int_0^{2\pi} d\phi \int_0^2 r\,dr\; r^2\cos^2\phi$$

$$= \underbrace{\int_0^{2\pi} d\phi\, \cos^2\phi}_{} \int_0^2 dr\, r^3 = \pi\left[\frac{1}{4}r^4\right]_{r=0}^{r=2} = 4\pi$$

$$= \frac{1}{2}\int_0^{2\pi} d\phi\,(\cos^2\phi + \sin^2\phi)$$

$$= \frac{1}{2}\int_0^{2\pi} d\phi\; 1 = \pi$$

(b) $f(\underline{r}) = \dfrac{x^2}{x^2+y^2} = \dfrac{r^2\cos^2\phi}{r^2(\cos^2\phi + \sin^2\phi)} = \cos^2\phi$

$$I = \int_0^{2\pi} d\phi \int_0^2 r\,dr\; \cos^2\phi = \pi\left[\frac{1}{2}r^2\right]_{r=0}^{r=2} = 2\pi$$

(c) $f(\underline{r}) = r$

$$I = \int_0^{2\pi} d\phi \int_0^2 r\,dr\; r = 2\pi\left[\frac{1}{3}r^3\right]_{r=0}^{r=2} = 2\pi\cdot\frac{8}{3} = \frac{16\pi}{3}$$

2.3 Volume Integrals

The concept of an area integral can be extended to *volume integrals*. The integration region is now a volume having some shape. The integration region is divided up into *volume elements* dV, and the function to be integrated $f(\mathbf{r})$ has a value in each element. Conceptually, the integral is the sum of all the contributions $f(\mathbf{r})\,dV$ over all the elements, in the limit where the volume elements become infinitesimally small.
In Cartesian coordinates, the volume element is simply $dV = dx\,dy\,dz$, and the volume integral can be expressed as a triple integral over x, y and z,

$$I = \int_V f(\mathbf{r})\,dV = \iiint_V f(x,y,z)\,dx\,dy\,dz. \tag{2.16}$$

The simplest case is when integration region is a cuboid. As an example, we want to calculate the volume integral of the function $f(\mathbf{r}) = \frac{1}{2}r^2 = \frac{1}{2}(x^2+y^2+z^2)$ over a cube whose 8 corners are at the points $(\pm1,\pm1,\pm1)$:

Write the limits explicitly:

$$I = \frac{1}{2}\int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz\,(x^2+y^2+z^2)\,.$$

Do z-integral first:

$$\begin{aligned} I &= \frac{1}{2}\int_{-1}^{1} dx \int_{-1}^{1} dy \left[x^2 z + y^2 z + \frac{1}{3}z^3\right]_{z=-1}^{z=1} \\ &= \int_{-1}^{1} dx \int_{-1}^{1} dy \left(x^2+y^2+\frac{1}{3}\right). \end{aligned}$$

Now do y-integral:

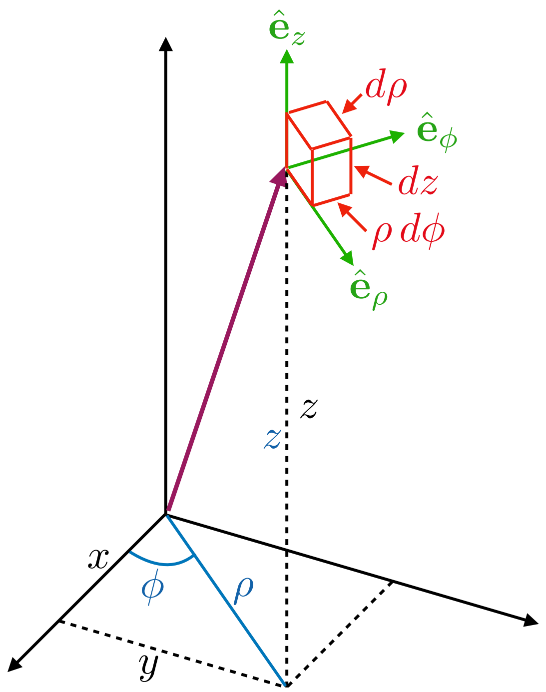
$$\begin{aligned} I &= \int_{-1}^{1} dx \left[x^2 y + \frac{1}{3}y^3 + \frac{1}{3}y\right]_{y=-1}^{y=1} \\ &= 2\int_{-1}^{1} dx \left(x^2 + \frac{2}{3}\right). \end{aligned}$$

And as a last step the x-integral

$$\begin{aligned} I &= 2\left[\frac{1}{3}x^3 + \frac{2}{3}x\right]_{-1}^{1} \\ &= 4. \end{aligned}$$

As in the case of area integrals, the concept of volume integrals can be applied to integration regions of arbitrary shape. Depending on the shape, the integrations could be much simpler in other coordinate systems such as cylindrical or spherical coordinates.

2.3.1 Cylindrical Coordinates



Cylindrical coordinates (ρ, ϕ, z) are just polar coordinates (ρ, ϕ) in the x-y-plane with z for the third variable,

$$x = \rho\, \cos\phi, \quad y = \rho\, \sin\phi, \quad z = z, \tag{2.17}$$

as illustrated in the figure above. Note that instead of ρ some books use r. We avoid this notation to make clear that ρ is different from the distance of the point $\mathbf{r} = (x, y, z)$ from the origin, which is $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2 + z^2}$.

From the transformation (2.3.1), it is straightforward to calculate the line element

$$\begin{aligned} d\mathbf{r} &= \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} d\rho\cos\phi - \rho\sin\phi\, d\phi \\ d\rho\sin\phi + \rho\cos\phi\, d\phi \\ dz \end{pmatrix} \\ &= \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix} d\rho + \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} \rho d\phi + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} dz. \end{aligned}$$

Hence

$$d\mathbf{r} = \hat{\mathbf{e}}_\rho d\rho + \hat{\mathbf{e}}_\phi \rho d\phi + \hat{\mathbf{e}}_z dz, \tag{2.18}$$

with

$$\hat{\mathbf{e}}_\rho = \begin{pmatrix} \cos\phi \\ \sin\phi \\ 0 \end{pmatrix}, \quad \hat{\mathbf{e}}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \quad \hat{\mathbf{e}}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.19}$$

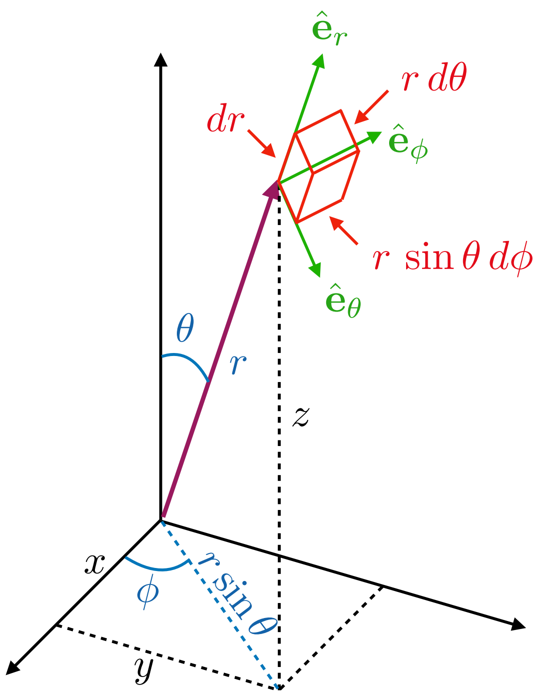
It is easy to check that $\hat{\mathbf{e}}_\rho$, $\hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_z$ are unit vectors that are orthogonal to one another. Changing the coordinates by infinitesimal amounts $d\rho$, $d\phi$ and dz defines an infinitesimal *volume element*

$$dV = d\rho \cdot \rho d\phi \cdot dz = \rho \, d\rho \, d\phi \, dz. \tag{2.20}$$

As an example, we calculate the volume of a cylinder with radius R and height H,

$$V = \int dV = \int_0^R \rho \, d\rho \int_0^H dz \int_0^{2\pi} d\phi = \frac{1}{2} R^2 \cdot H \cdot 2\pi = \pi R^2 H.$$

2.3.2 Spherical Coordinates



Each point $\mathbf{r} = (x, y, z)$ is uniquely determined by its distance r from the origin, the angle θ the vector $\mathbf{r}$ forms with the z-axis, and the angle ϕ between the projection of $\mathbf{r}$ into the x-y-plane and the x-axis (see above figure). The transformation between spherical coordinates (r, θ, ϕ) and cartesian coordinates (x, y, z) is given by

$$x = r\,\sin\theta\,\cos\phi, \quad y = r\,\sin\theta\,\sin\phi, \quad z = r\,\cos\theta. \tag{2.21}$$

We proceed to compute the line element in spherical coordinates:

$$\begin{aligned} d\mathbf{r} &= \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \\ &= \begin{pmatrix} \sin\theta\,\cos\phi \\ \sin\theta\,\sin\phi \\ \cos\theta \end{pmatrix} dr + \begin{pmatrix} \cos\theta\,\cos\phi \\ \cos\theta\,\sin\phi \\ -\sin\theta \end{pmatrix} r d\theta + \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix} r\sin\theta d\phi. \end{aligned}$$

Hence

$$d\mathbf{r} = \hat{\mathbf{e}}_r dr + \hat{\mathbf{e}}_\theta r d\theta + \hat{\mathbf{e}}_\phi r\sin\theta d\phi, \tag{2.22}$$

with

$$\hat{\mathbf{e}}_r = \begin{pmatrix} \sin\theta\,\cos\phi \\ \sin\theta\,\sin\phi \\ \cos\theta \end{pmatrix}, \quad \hat{\mathbf{e}}_\theta = \begin{pmatrix} \cos\theta\,\cos\phi \\ \cos\theta\,\sin\phi \\ -\sin\theta \end{pmatrix}, \quad \hat{\mathbf{e}}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{2.23}$$

The vectors $\hat{\mathbf{e}}_r$, $\hat{\mathbf{e}}_\theta$ and $\hat{\mathbf{e}}_\phi$ form a set of orthogonal unit vectors. As illustrated in the figure, the volume element in spherical coordinates is given by

$$dV = dr \cdot r d\theta \cdot r\sin\theta d\phi = r^2 dr\,\sin\theta d\theta\,d\phi. \tag{2.24}$$

As a simple example, we calculate the volume of a sphere of radius R,

$$V = \int_0^R r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi = \left[\frac{1}{3}r^3\right]_0^R \cdot \Big[-\cos\theta\Big]_0^\pi \cdot 2\pi = \frac{4}{3}\pi R^3.$$

2.3.3 Worked Examples

(1) Use spherical coordinates to evaluate the volume integral

$$I = \int (x^2 + y^2)\, dV$$

over the interior of the sphere of radius 1 centred at the origin.

(2) Use cylindrical coordinates to calculate the value of the volume integral

$$I = \int (x^2 + y^2) z^2\, dV,$$

where the integration region is the interior of the cylinder of radius 1 and length 2 centred on the origin, and the axis of the cylinder coincides with the z-axis.

(3) Calculate the volume of a cone with radius R and hight H.

1) $I = \int dV\, (x^2+y^2)$

$$x^2+y^2 = r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi = r^2 \sin^2\theta$$

$$dV = r^2\, dr\, \sin\theta\, d\theta\, d\phi$$

$$\rightarrow I = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin^3\theta \int_0^1 dr\, r^4$$

$$= 2\pi \cdot \frac{1}{5} \int_0^{\pi} d\theta\, \sin\theta\, (1-\cos^2\theta)$$

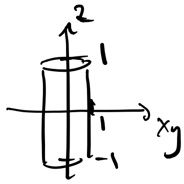
$$\underset{\substack{u = \cos\theta \\ du = -\sin\theta\, d\theta}}{=} -\frac{2}{5}\pi \int_1^{-1} du\, (1-u^2) = -\frac{2}{5}\pi \left[u - \frac{1}{3}u^3\right]_{u=1}^{u=-1}$$

$$= \frac{4}{5}\pi\left(1 - \frac{1}{3}\right) = \frac{8}{15}\pi$$

2) $I = \int dV \, (x^2+y^2) z^2$

$dV = \rho \, d\rho \, d\phi \, dz$

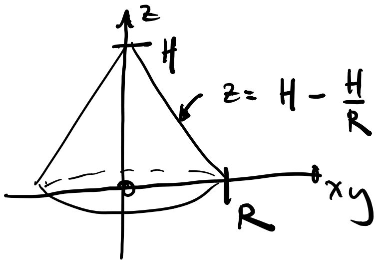
$(x^2+y^2) z^2 = \rho^2 z^2$



$I = \int_0^{2\pi} d\phi \int_0^1 d\rho \, \rho^3 \int_{-1}^{1} dz \, z^2$

$= 2\pi \cdot \frac{1}{4} \cdot \frac{2}{3} = \frac{\pi}{3}.$

3)



use cylindrical coordinates, $dV = \rho \, d\rho \, d\phi \, dz$

$\phi \in [0, 2\pi)$

$\rho \in [0, R] \rightarrow z \in [0, H - \frac{H}{R} \cdot \rho]$

$V = \int_0^{2\pi} d\phi \int_0^R d\rho \, \rho \int_0^{H - \frac{H}{R}\rho} dz$

$= 2\pi \int_0^R d\rho \, \rho \left(H - \frac{H}{R}\rho\right) = 2\pi \left[\frac{1}{2} H \rho^2 - \frac{1}{3} \frac{H}{R} \rho^3\right]_{\rho=0}^{\rho=R}$

$= 2\pi \left(\frac{1}{2} H R^2 - \frac{1}{3} H R^2\right) = \frac{1}{3} \pi R^2 H$

2.4 Surface Integrals

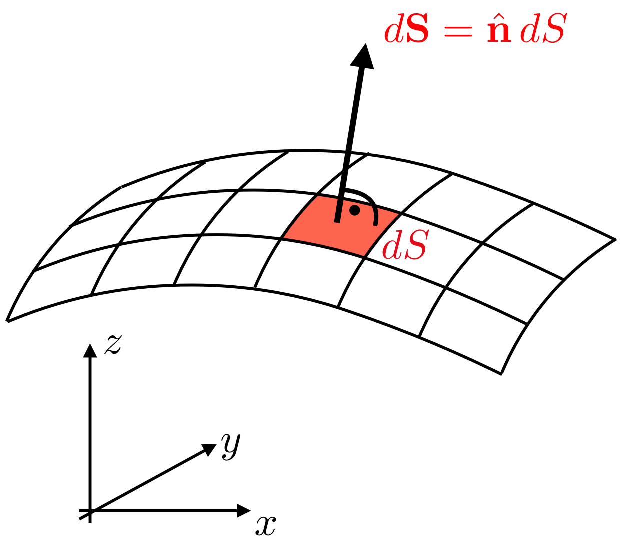
In Sec. 2.2, we have introduced the area integral over two-dimensional regions in the x-y plane. Here we will generalise this to integrals over *(curved) surfaces* S embedded in three-dimensional space. For any scalar field $f(\mathbf{r}) = f(x, y, z)$ that is defined on all the points of the surface S we can define the *surface integral* by by dividing S into infinitesimally small surface area elements dS and summing all the individual contributions $f(\mathbf{r})\, dS$ in the limit that $dS \to 0$,

$$I_1 = \int_S f(\mathbf{r})\, dS. \tag{2.25}$$

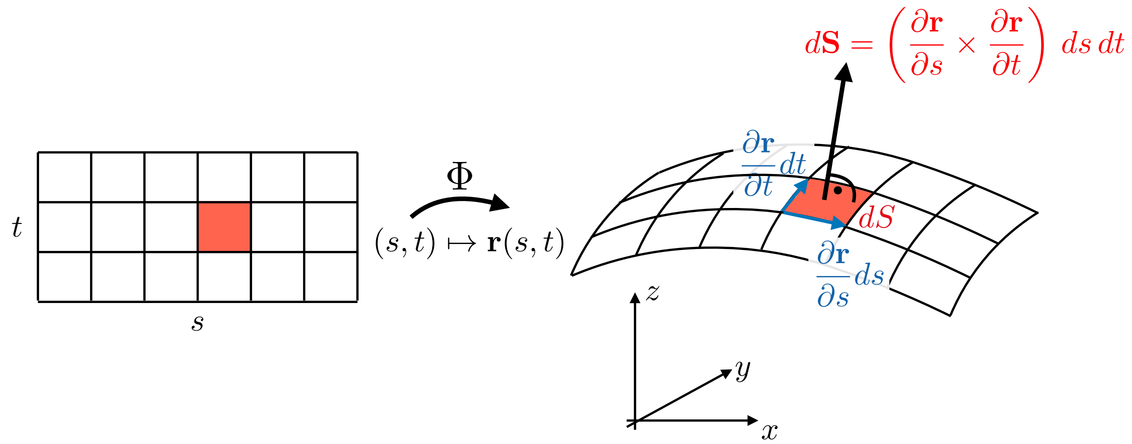
We would have to compute such an integral if we wanted to calculate the total charge of a curved surface for a given charge density. In physics we often need to calculate the total *flux* of a vector field $\mathbf{G}(\mathbf{r})$ through a surface. For example, $\mathbf{G}(\mathbf{r})$ could be a current density and we would like to compute the total current through S. The idea is to decompose $\mathbf{G}$ into the components parallel and perpendicular to S. Only the perpendicular component contributes to the flux. It can be written as $G_\perp = \mathbf{G} \cdot \hat{\mathbf{n}}$, where $\hat{\mathbf{n}} = \hat{\mathbf{n}}(\mathbf{r})$ is a unit vector normal (orthogonal) to the surface at the point $\mathbf{r}$. The total flux is therefore given by the surface integral

$$I_2 = \int_S (\mathbf{G} \cdot \hat{\mathbf{n}})\, dS = \int_S \mathbf{G} \cdot d\mathbf{S}, \tag{2.26}$$

where in the last step we have defined the *vectorial surface element* $d\mathbf{S} = \hat{\mathbf{n}}\, dS$ normal to S, as illustrated below.



In order to calculate such surface integrals we first need to *parametrise the surface*. As for the line integral, the result of a surface integral does not depend on the choice of the parametrisation. Since surfaces are two-dimensional objects, surface integrals are double integrals. Let us first derive general expressions for surface integrals of the form (2.25) or (2.26), assuming that S is parametrised by $\mathbf{r} = \mathbf{r}(s,t)$ with $s \in [s_{\min}, s_{\max}]$ and $t \in [t_{\min}, t_{\max}]$. We can easily express the fields as functions of s and t, using that $\mathbf{r} = \mathbf{r}(s,t)$. But how to calculate $d\mathbf{S}$ and dS?



To obtain the surface area element we need to calculate the infinitesimal changes of the vector $\mathbf{r}(s,t)$ as we change the coordinates s and t by infinitesimal amounts ds and dt, respectively. This defines infinitesimally small vectors

$$d\mathbf{u} = \frac{\partial \mathbf{r}}{\partial s}\, ds \quad \text{and} \quad d\mathbf{v} = \frac{\partial \mathbf{r}}{\partial t}\, dt$$

tangential to the surface S. The surface area element dS is equal to the area of the parallelogram formed by the two vectors $d\mathbf{u}$ and $d\mathbf{v}$. $d\mathbf{S}$ is the vector of length dS that is perpendicular to the two tangential vectors $d\mathbf{u}$ and $d\mathbf{v}$. We can therefore calculate the vectorial surface-area element form the vector product (cross product)

$$d\mathbf{S} = d\mathbf{u} \times d\mathbf{v} = \left(\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right) ds\, dt, \tag{2.27}$$

leading to the final results

$$I_1 = \int_{s_{\min}}^{s_{\max}} ds \int_{t_{\min}}^{t_{\max}} dt\ f\big(\mathbf{r}(s,t)\big) \left|\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right|, \tag{2.28a}$$

$$I_2 = \int_{s_{\min}}^{s_{\max}} ds \int_{t_{\min}}^{t_{\max}} dt\ \mathbf{G}\big(\mathbf{r}(s,t)\big) \cdot \left(\frac{\partial \mathbf{r}}{\partial s} \times \frac{\partial \mathbf{r}}{\partial t}\right). \tag{2.28b}$$

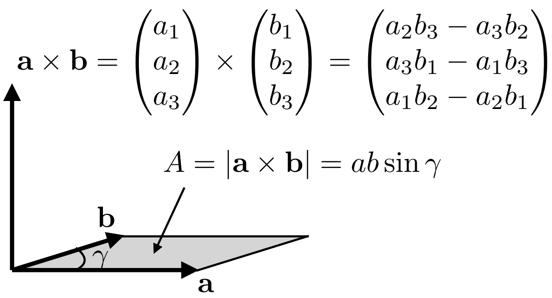


Figure 2.5: Illustration and definition of the vector or cross product $\mathbf{a} \times \mathbf{b}$. The resulting vector is perpendicular to $\mathbf{a}$ and $\mathbf{b}$ and its length $|\mathbf{a} \times \mathbf{b}|$ is equal to the area of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$.

2.4.1 Surface Integrals in Spherical Coordinates

If the surface is part of a sphere of a fixed radius R the natural choice is to use spherical coordinates and to parametrise the surface by the angles (θ, ϕ),

$$\mathbf{r} = \mathbf{r}(\theta, \phi) = R \begin{pmatrix} \sin\theta\,\cos\phi \\ \sin\theta\,\sin\phi \\ \cos\theta \end{pmatrix}. \tag{2.29}$$

We already know that

$$d\mathbf{S} = R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{e}}_r, \tag{2.30}$$

since $dV = dS\, dr$ and since $\hat{\mathbf{e}}_r$ is the unit vector normal to the sphere. However, let's calculate the area element from the general expression (2.27),

$$\begin{aligned} d\mathbf{S} &= \left(\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial \phi}\right) d\theta\, d\phi \\ &= R^2 \begin{pmatrix} \cos\theta\,\cos\phi \\ \cos\theta\,\sin\phi \\ -\sin\theta \end{pmatrix} \times \begin{pmatrix} -\sin\theta\,\sin\phi \\ \sin\theta\,\cos\phi \\ 0 \end{pmatrix} d\theta\, d\phi \\ &= R^2 \begin{pmatrix} \sin^2\theta\,\cos\phi \\ \sin^2\theta\,\sin\phi \\ \sin\theta\,\cos\theta \end{pmatrix} d\theta\, d\phi \\ &= R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{e}}_r. \end{aligned}$$

As an example, we compute the surface area of a sphere of radius R,

$$A = \int_{\text{sphere}} dS = R^2 \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\, d\theta = 2\pi R^2 \left[-\cos\theta\right]_{\theta=0}^{\theta=\pi} = 4\pi R^2.$$

2.4.2 Surfaces Defined by $z = g(x, y)$

Often the surface is defined by a function that relates the z coordinate to the x and y coordinates, $z = g(x, y)$. E.g., the equation $z = \sqrt{R^2 - x^2 - y^2}$ describes an upper-half sphere of radius R. In such cases we can paramtrise the surface by the coordinates (x, y),

$$\mathbf{r} = \mathbf{r}(x, y) = \begin{pmatrix} x \\ y \\ g(x, y) \end{pmatrix}. \tag{2.31}$$

From this parametrisation we obtain the vectorial surface are element,

$$d\mathbf{S} = \left(\frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right) dx\, dy = \begin{pmatrix} 1 \\ 0 \\ \partial g/\partial x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \partial g/\partial y \end{pmatrix} dx\, dy = \begin{pmatrix} -\partial g/\partial x \\ -\partial g/\partial y \\ 1 \end{pmatrix} dx\, dy,$$

and therefore

$$d\mathbf{S} = \begin{pmatrix} -\partial g/\partial x \\ -\partial g/\partial y \\ 1 \end{pmatrix} dx\, dy, \tag{2.32}$$

$$dS = \sqrt{1 + \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2}\, dx\, dy. \tag{2.33}$$

2.4.3 Worked Examples

(1) Calculate the surface integral $I = \int_S \mathbf{G}(\mathbf{r}) \cdot d\mathbf{S}$ for the vector field $\mathbf{G}(\mathbf{r}) = z\hat{\mathbf{e}}_x + x\hat{\mathbf{e}}_z$ and the surface that is determined by $z = x^2 + y$ for $x \in [0,1]$ and $y \in [0,1]$.

(2) Use cylindrical coordinates to calculate the area of the surface given by $z = 1 - x^2 - y^2$ for $x^2 + y^2 \leq 1$.

(3) Given is the sphere $x^2 + y^2 + z^2 \leq 4$ and the plane $z = 1$ parallel to the xy-plane.

- (a) Calculate the surface area of the cap of the sphere that is above the plane ($z > 1$).
- (b) Use spherical coordinates to calculate the volume of the part of the sphere that is above the plane.
- (c) Calculate the same volume using cylindrical coordinates.

(1) $\underline{r}(x,y) = \begin{pmatrix} x \\ y \\ x^2+y \end{pmatrix} \qquad \underline{G}(\underline{r}(x,y)) = \begin{pmatrix} x^2+y \\ 0 \\ x \end{pmatrix}$

$$d\underline{S} = \left(\frac{\partial \underline{r}}{\partial x} \times \frac{\partial \underline{r}}{\partial y}\right) dx\,dy$$

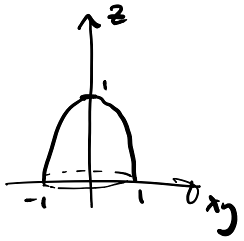
$$= \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} dx\,dy = \begin{pmatrix} -2x \\ -1 \\ 1 \end{pmatrix} dx\,dy$$

$$I = \int_0^1 dx \int_0^1 dy \begin{pmatrix} x^2+y \\ 0 \\ x \end{pmatrix} \cdot \begin{pmatrix} -2x \\ -1 \\ 1 \end{pmatrix}$$

$$= \int_0^1 dx \int_0^1 dy \,(-2x^3 - 2xy + x)$$

$$= \int_0^1 dx \,(-2x^3 - x + x) = \left[-\tfrac{1}{2}x^4\right]_0^{x=1} = -\tfrac{1}{2}$$

(2)



$$x = \rho\cos\phi\,,\quad y = \rho\sin\phi$$

$$z = 1 - (x^2 + y^2) = 1 - \rho^2$$

$$\underline{r}(\rho, \phi) = \begin{pmatrix} \rho\cos\phi \\ \rho\sin\phi \\ 1-\rho^2 \end{pmatrix}$$

$$\frac{\partial \underline{r}}{\partial \rho} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ -2\rho \end{pmatrix} \qquad \frac{\partial \underline{r}}{\partial \phi} = \begin{pmatrix} -\rho\sin\phi \\ \rho\cos\phi \\ 0 \end{pmatrix}$$

$$\Rightarrow d\underline{S} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ -2\rho \end{pmatrix} \times \begin{pmatrix} -\rho\sin\phi \\ \rho\cos\phi \\ 0 \end{pmatrix} d\rho\, d\phi$$

$$= \begin{pmatrix} 2\rho^2\cos\phi \\ 2\rho^2\sin\phi \\ \rho \end{pmatrix} d\rho\, d\phi = \begin{pmatrix} 2\rho\cos\phi \\ 2\rho\sin\phi \\ 1 \end{pmatrix} \rho\, d\rho\, d\phi$$

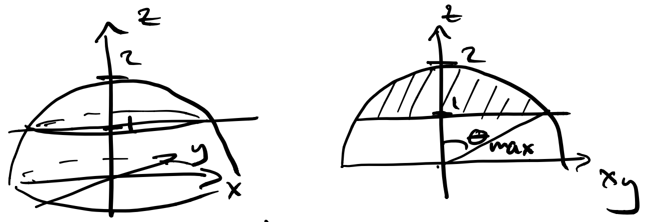
$$\Rightarrow dS = \sqrt{1+4\rho^2}\ \rho\, d\rho\, d\phi$$

$$A = \int_0^{2\pi} d\phi \int_0^1 d\rho\, \rho\sqrt{1+4\rho^2} \qquad u = 1+4\rho^2,\quad du = 8\rho\, d\rho$$

$$= 2\pi \cdot \frac{1}{8}\int_1^5 du\, u^{1/2}$$

$$= \frac{\pi}{4}\cdot\frac{2}{3}\, u^{3/2}\Big|_{u=1}^{u=5} = \frac{\pi}{6}\left(\sqrt{5}^3 - 1\right)$$

(3) (a)



$\phi \in [0, 2\pi]$ $\quad \theta \in [0, \theta_{max}]$ where $\cos\theta_{max} = \frac{1}{2}$

$\Rightarrow \theta_{max} = \pi/3$

$$A = \int dS = 4 \int_0^{\pi/3} d\theta \sin\theta \int_0^{2\pi} d\phi$$

$\uparrow$

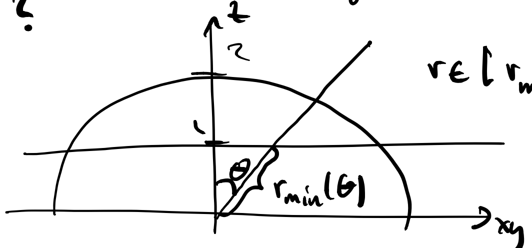
$dS = R^2 \sin\theta \, d\theta \, d\phi$

$$= 4 \left[-\cos\theta \right]_0^{\pi/3} \cdot 2\pi = 4\pi$$

(b) $V = \int dV \qquad dV = r^2 dr \sin\theta d\theta d\phi$

$\phi \in [0, 2\pi]$, $\theta \in [0, \pi/3]$

what are the integration limits for r ?



$r \in [r_{min}(\theta), 2]$

$$\cos\theta = \frac{1}{r_{min}(\theta)} \Rightarrow r_{min}(\theta) = \frac{1}{\cos\theta}$$

$$V = \int_0^{2\pi} d\phi \int_0^{\pi/3} d\theta \, \sin\theta \int_{\frac{1}{\cos\theta}}^{2} dr \, r^2$$

$$= 2\pi \int_0^{\pi/3} d\theta \, \sin\theta \left[\frac{1}{3} r^3 \right]_{r=\frac{1}{\cos\theta}}^{r=2}$$

$$= \frac{2\pi}{3} \int_0^{\pi/3} d\theta \, \sin\theta \left(8 - \frac{1}{\cos^3\theta} \right)$$

$$\underset{\substack{\xi = \cos\theta \\ d\xi = -\sin\theta \, d\theta}}{=} -\frac{2\pi}{3} \int_1^{1/2} d\xi \, (8 - \xi^{-3})$$

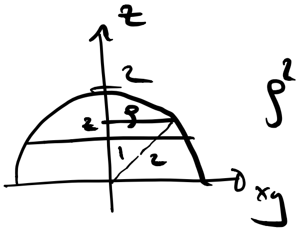
$$= -\frac{2\pi}{3} \left[8\xi + \frac{1}{2} \xi^{-2} \right]_{\xi=1}^{\xi=1/2}$$

$$= -\frac{2\pi}{3} \left(4 + 2 - 8 - \frac{1}{2} \right)$$

$$= \frac{2\pi}{3} \cdot \frac{5}{2} = \frac{5}{3} \pi$$

(c) cylindrical coordinates : $dV = \rho \, d\rho \, d\phi \, dz$

$\phi \in [0, 2\pi]$, $z \in [1, 2]$



$\rho^2 + z^2 = 4 \Rightarrow \rho = \sqrt{4 - z^2}$

$$V = \int_0^{2\pi} d\phi \int_1^2 dz \int_0^{\sqrt{4-z^2}} d\rho \, \rho$$

$$= 2\pi \int_1^2 dz \left[\frac{1}{2} \rho^2 \right]_0^{\sqrt{4-z^2}}$$

$$= \pi \int_1^2 dz \, (4 - z^2)$$

$$= \pi \left[4z - \frac{1}{3} z^3 \right]_1^2$$

$$= \pi \left[8 - \frac{8}{3} - \left(4 - \frac{1}{3} \right) \right]$$

$$= \pi \left(4 - 7/3 \right) = \frac{5}{3} \pi$$

2.5 Gauss's Divergence Theorem

So far we have discussed different classes of multidimensional integrals: line, area, volume and surface integrals. We have seen that surface integrals can be difficult to compute since surfaces are *curved* two-dimensional manifolds embedded in three-dimensional space.

Gauss's Divergence Theorem (or short: *Gauss's Theorem*) relates the surface integral over a *closed* surface S to a volume integral over the volume V enclosed by the surface. In this case, S is the *boundary* of V, which we denote by $S = \partial V$.

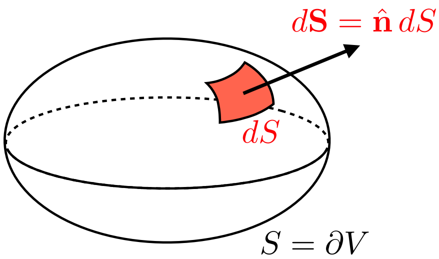
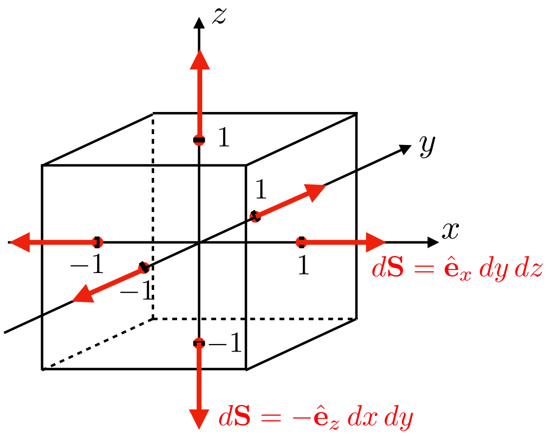


Figure 2.6: Vectorial surface-area element $d\mathbf{S}$ of a closed surface $S = \partial V$. The convention is that $d\mathbf{S}$ points outwards.

Gauss's theorem states that the flux of a vector field $\mathbf{G}(\mathbf{r})$ through a closed surface $S = \partial V$ is equal to the integral of the divergence of $\mathbf{G}$ over the enclosed volume V,

$$\int_{S=\partial V} \mathbf{G}(\mathbf{r}) \cdot d\mathbf{S} = \int_V \mathrm{div}\mathbf{G}\; dV. \tag{2.34}$$

Let us check the theorem for a simple example: we calculate the flux of the vector field $\mathbf{G}(\mathbf{r}) = x^2\, \hat{\mathbf{e}}_x + y\, \hat{\mathbf{e}}_y + z\, \hat{\mathbf{e}}_z$ through a cube with corners $(\pm 1, \pm 1, \pm 1)$.



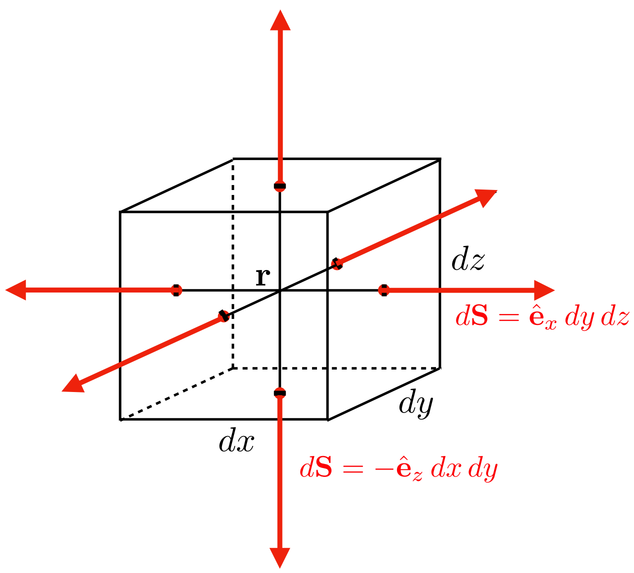
$$
\begin{aligned}
I = \int_{S=\partial V} \mathbf{G}(\mathbf{r})\cdot d\mathbf{S} \;&=\; \int_{-1}^{1} dy \int_{-1}^{1} dz \begin{pmatrix}1\\ y\\ z\end{pmatrix}\cdot\begin{pmatrix}1\\ 0\\ 0\end{pmatrix} + \int_{-1}^{1} dy \int_{-1}^{1} dz \begin{pmatrix}1\\ y\\ z\end{pmatrix}\cdot\begin{pmatrix}-1\\ 0\\ 0\end{pmatrix}\\
&\quad + \int_{-1}^{1} dx \int_{-1}^{1} dz \begin{pmatrix}x^2\\ 1\\ z\end{pmatrix}\cdot\begin{pmatrix}0\\ 1\\ 0\end{pmatrix} + \int_{-1}^{1} dx \int_{-1}^{1} dz \begin{pmatrix}x^2\\ -1\\ z\end{pmatrix}\cdot\begin{pmatrix}0\\ -1\\ 0\end{pmatrix}\\
&\quad + \int_{-1}^{1} dx \int_{-1}^{1} dy \begin{pmatrix}x^2\\ y\\ 1\end{pmatrix}\cdot\begin{pmatrix}0\\ 0\\ 1\end{pmatrix} + \int_{-1}^{1} dx \int_{-1}^{1} dy \begin{pmatrix}x^2\\ y\\ -1\end{pmatrix}\cdot\begin{pmatrix}0\\ 0\\ -1\end{pmatrix}\\
&=\; 2\int_{-1}^{1} dx \int_{-1}^{1} dz + 2\int_{-1}^{1} dx \int_{-1}^{1} dy = 16.
\end{aligned}
$$

We now calculate the integral using Gauss's theorem:

$$
\begin{aligned}
I &= \int_V (\nabla\cdot\mathbf{G})dV = \int_{-1}^{1} dx \int_{-1}^{1} dy \int_{-1}^{1} dz\;(2x+2)\\
&= 4\int_{-1}^{1} dx\;(2x+2) = 4\left[x^2+2x\right]_{-1}^{1} = 16,
\end{aligned}
$$

in agreement with the previous result.

A formal mathematical proof of Gauss's theorem is lengthy and tedious. Instead we provide a sketch of the proof which gives the main idea. As a first step, we prove the theorem for an infinitesimal cuboid of volume $dV = dx\,dy\,dz$, centred around a point $\mathbf{r}$ in the volume V.



The infinitesimal flux dF through the surface of the cube is given by

$$\begin{aligned} dF &= \mathbf{G}\left(\mathbf{r}+\frac{dx}{2}\hat{\mathbf{e}}_x\right)\cdot\hat{\mathbf{e}}_x\,dy\,dz+\mathbf{G}\left(\mathbf{r}-\frac{dx}{2}\hat{\mathbf{e}}_x\right)\cdot(-\hat{\mathbf{e}}_x)\,dy\,dz \\ &\quad+\mathbf{G}\left(\mathbf{r}+\frac{dy}{2}\hat{\mathbf{e}}_y\right)\cdot\hat{\mathbf{e}}_y\,dy\,dz+\mathbf{G}\left(\mathbf{r}-\frac{dy}{2}\hat{\mathbf{e}}_y\right)\cdot(-\hat{\mathbf{e}}_y)\,dy\,dz \\ &\quad+\mathbf{G}\left(\mathbf{r}+\frac{dz}{2}\hat{\mathbf{e}}_z\right)\cdot\hat{\mathbf{e}}_z\,dx\,dy+\mathbf{G}\left(\mathbf{r}-\frac{dz}{2}\hat{\mathbf{e}}_z\right)\cdot(-\hat{\mathbf{e}}_z)\,dx\,dy. \end{aligned}$$

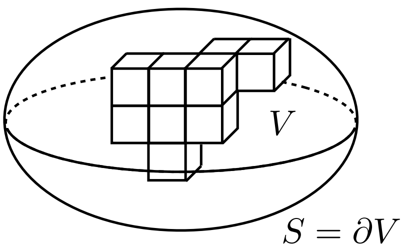
Let us focus on the first line and simplify further:

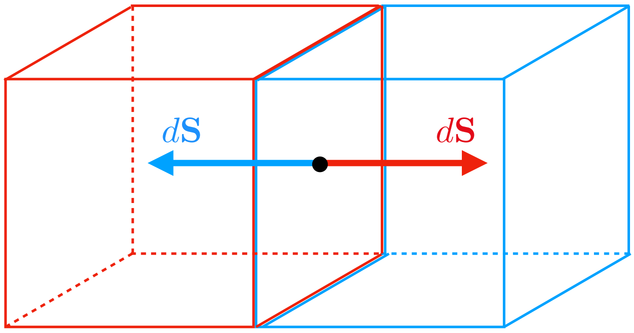
$$\begin{aligned} &\left[\mathbf{G}\left(\mathbf{r}+\frac{dx}{2}\hat{\mathbf{e}}_x\right)\cdot\hat{\mathbf{e}}_x+\mathbf{G}\left(\mathbf{r}-\frac{dx}{2}\hat{\mathbf{e}}_x\right)\cdot(-\hat{\mathbf{e}}_x)\right]\,dy\,dz \\ =&\left[G_x\left(\mathbf{r}+\frac{dx}{2}\hat{\mathbf{e}}_x\right)-G_x\left(\mathbf{r}-\frac{dx}{2}\hat{\mathbf{e}}_x\right)\right]\,dy\,dz \\ =&\left[G_x(\mathbf{r})+\partial_x G_x(\mathbf{r})\frac{dx}{2}-\left(G_x(\mathbf{r})-\partial_x G_x(\mathbf{r})\frac{dx}{2}\right)\right]\,dy\,dz \\ =&\ \partial_x G_x(\mathbf{r})\,dx\,dy\,dz, \end{aligned}$$

and therefore

$$dF = (\partial_x G_x + \partial_y G_y + \partial_z G_z)\ dx\,dy\,dz = \operatorname{div}\mathbf{G}\ dV. \tag{2.35}$$

We now consider the integral over a volume V with surface $S = \partial V$. The integral is defined as a "sum" over little cuboids dV. Integrating the r.h.s. of Eq. (2.35) we obtain $\int_V \operatorname{div}\mathbf{G}\,dV$.





Summing the fluxes $dF = \mathbf{G}\cdot d\mathbf{S}$ through the surfaces of all cuboids we realise that the contributions from shared interfaces vanish since $d\mathbf{S}$ for the face-sharing, adjacent cuboids are anti-parallel.
Only the contributions from the surface survive,

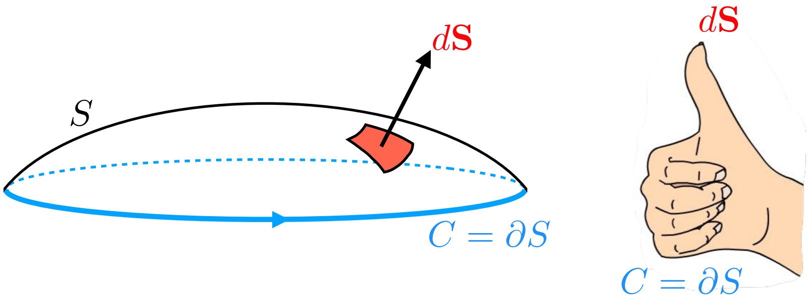
$$\int_V dF = \int_{S=\partial V} dF = \int_{S=\partial V} \mathbf{G}\cdot d\mathbf{S}.$$

2.6 Stokes's Theorem

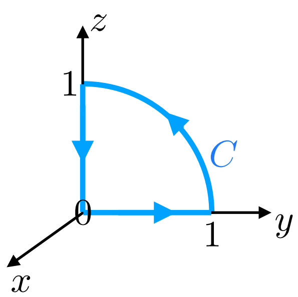
Stokes's theorem states that the *loop integral* of a vector field $\mathbf{G}(\mathbf{r})$ around the boundary $C = \partial S$ of an *open* surface S is equal to the *flux* of the curl of the vector field, $\mathrm{curl}\,\mathbf{G} = \nabla \times \mathbf{G}$ through the surface,

$$\oint_{C=\partial S} \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r} = \int_S \mathrm{curl}\,\mathbf{G} \cdot d\mathbf{S}, \tag{2.36}$$

where the relative orientation of the line integral C and $d\mathbf{S}$ satisfies the right-hand rule, as illustrated in the figure.[2]



As an example we calculate the line integral of the vector field $\mathbf{G}(\mathbf{r}) = y\,\hat{\mathbf{e}}_x - z\,\hat{\mathbf{e}}_y - x^2\,\hat{\mathbf{e}}_z$ over the closed path C shown in the following figure:



The result for the line integral is independent of the parametrisation we chose and loop integrals are independent of the starting position. We start at the origin and parametrise the three pieces of the path as follows:

$$C_1 : \ \mathbf{r}(t) = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}, \quad C_2 : \ \mathbf{r}(t) = \begin{pmatrix} 0 \\ \cos(t\pi/2) \\ \sin(t\pi/2) \end{pmatrix}, \quad C_3 : \ \mathbf{r}(t) = \begin{pmatrix} 0 \\ 0 \\ 1-t \end{pmatrix},$$

with $t \in [0, 1]$ for all three pieces. All contributions are of the form

[2]Note that this statement is strictly true only for a flat surface (constant $d\mathbf{S}$). For a curved surface as shown in the figure the statement is true for the average $d\mathbf{S}$.

$$I = \int_0^1 dt\, \mathbf{G}(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Using the parametrisation above we obtain

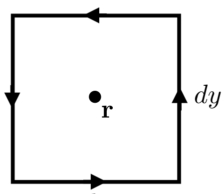
$$\begin{aligned} I &= \int_0^1 dt \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \int_0^1 dt \begin{pmatrix} \cos(t\pi/2) \\ -\sin(t\pi/2) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\pi/2\sin(t\pi/2) \\ \pi/2\cos(t\pi/2) \end{pmatrix} \\ &\quad + \int_0^1 dt \begin{pmatrix} 0 \\ -1+t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\ &= \frac{\pi}{2} \int_0^1 dt\, \sin^2(t\pi/2) = \frac{\pi}{4} \int_0^1 dt\, [1 - \cos(t\pi)] = \frac{\pi}{4} \left[t - \frac{1}{\pi} \sin(t\pi) \right]_0^1 = \frac{\pi}{4}. \end{aligned}$$

We now compute the integral using Stokes's theorem. Here the surface S is not given and there are infinitely many surfaces with $C = \partial S$. The simplest choice is to use the flat surface in the y-z-plane (area of a quarter circle of radius 1). In this case the vectorial surface area element is constant and given by $d\mathbf{S} = \hat{\mathbf{e}}_x\, dy\, dz$. Note that for this orientation of $d\mathbf{S}$ the right-hand rule is satisfied. We obtain

$$I = \int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S} = \int_S \begin{pmatrix} 1 \\ 2x \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dy\, dz = \int_S dy\, dz = \frac{\pi}{4},$$

which is simply the area of a quarter of a unit circle and equal to the loop integral calculated above.

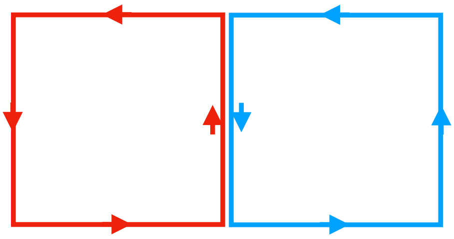
Let us conclude this section with a sketch of a proof of Stokes's theorem. We first prove the theorem for an infinitesimal surface area element. Without loss of generality, we can use a local coordinate system for which the patch is in the x-y plane and the vectorial surface-are element given by $d\mathbf{S} = \hat{\mathbf{e}}_z\, dx\, dy$,



The loop "integral" of a vector field $\mathbf{G}(\mathbf{r})$ around the infinitesimal patch around $\mathbf{r}$ is simply given by the sum of the contributions from the four sides,

$$
\begin{aligned}
dI &= \mathbf{G}\left(\mathbf{r} - \frac{dy}{2}\hat{\mathbf{e}}_y\right) \cdot \hat{\mathbf{e}}_x \, dx + \mathbf{G}\left(\mathbf{r} + \frac{dx}{2}\hat{\mathbf{e}}_x\right) \cdot \hat{\mathbf{e}}_y \, dy \\
&\quad + \mathbf{G}\left(\mathbf{r} + \frac{dy}{2}\hat{\mathbf{e}}_y\right) \cdot (-\hat{\mathbf{e}}_x) \, dx + \mathbf{G}\left(\mathbf{r} - \frac{dx}{2}\hat{\mathbf{e}}_x\right) \cdot (-\hat{\mathbf{e}}_y) \, dy \\
&= \left[G_x\left(\mathbf{r} - \frac{dy}{2}\hat{\mathbf{e}}_y\right) - G_x\left(\mathbf{r} + \frac{dy}{2}\hat{\mathbf{e}}_y\right)\right] dx + \left[G_y\left(\mathbf{r} + \frac{dx}{2}\hat{\mathbf{e}}_x\right) - G_y\left(\mathbf{r} - \frac{dx}{2}\hat{\mathbf{e}}_x\right)\right] dy \\
&= -\partial_y G_x(\mathbf{r}) \, dx \, dy + \partial_x G_y(\mathbf{r}) \, dx \, dy \\
&= (\nabla \times \mathbf{G})_z \, dx \, dy = (\nabla \times \mathbf{G}) \cdot \hat{\mathbf{e}}_z \, dx \, dy = (\nabla \times \mathbf{G}) \cdot d\mathbf{S}.
\end{aligned}
$$

We integrate both sides of this equation over the surface S by summing the contributions for all surface-area elements that make up S. The r.h.s. immediately gives $\int_S (\nabla \times \mathbf{G}) \cdot d\mathbf{S}$. Summing the loop integral contributions from all the patches we immediately realise that all internal contributions vanish due to a cancelation from neighbouring patches:



Only the contributions from the boundary $C = \partial S$ of S survive,

$$
\int_S dI = \int_{C=\partial S} \mathbf{G} \cdot d\mathbf{r}.
$$

2.7 Worked Examples

(1) Calculate the flux of the vector field $\mathbf{G}(\mathbf{r}) = x\,\hat{\mathbf{e}}_x$ through a unit sphere around the origin, using a surface integral. Then apply Gauss's theorem to obtain the same result.

Spherical coordinates:

$$d\underline{S} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{\underline{e}}_r \underset{R=1}{=} \sin\theta \, d\theta \, d\phi \, \hat{\underline{e}}_r$$

$\theta \in [0, \pi]$; $\phi \in [0, 2\pi]$

$$\hat{\underline{e}}_r = \begin{pmatrix} \sin\theta \cos\phi \\ \sin\theta \sin\phi \\ \cos\theta \end{pmatrix} ; \quad \underline{G}(\underline{r}) = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi \\ 0 \\ 0 \end{pmatrix}$$

$$I = \int_{S=\partial V} \underline{G}(\underline{r}) \cdot d\underline{S}$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \, \sin\theta \, \sin^2\theta \cos^2\phi$$

$$= \underbrace{\int_0^{2\pi} d\phi \cos^2\phi}_{=\pi} \; \underbrace{\int_0^{\pi} d\theta \, \sin\theta (1 - \cos^2\theta)}_{\underset{\xi = \cos\theta}{=} \int_{-1}^{1} d\xi \, (1-\xi^2) = 2\left[\xi - \frac{1}{3}\xi^3\right]_0^1}$$

$$= \frac{4}{3}\pi$$

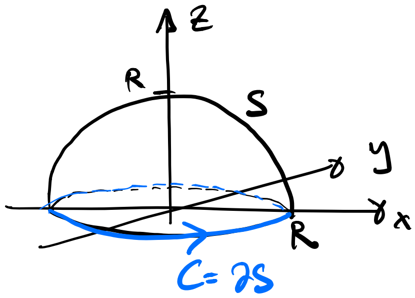
Now using Gauss's theorem:

$$I = \int_V \underbrace{\operatorname{div} \underline{G}}_{=1} \, dV = \int_V dV = \frac{4}{3}\pi \quad \text{(volume of unit sphere)}$$

(2) Consider the surface S defined by $x^2 + y^2 + z^2 = R^2$ and $z \geq 0$ and the vector field $\mathbf{G}(\mathbf{r}) = -y\,\hat{\mathbf{e}}_x + x\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z$. First calculate the line integral

$$I = \oint_{C=\partial S} \mathbf{G}(\mathbf{r}) \cdot d\mathbf{r},$$

then obtain the same result by using Stokes's theorem and evaluating the resulting surface integral.



$$\underline{G}(\underline{r}) = \begin{pmatrix} -y \\ x \\ z \end{pmatrix}$$

Line integral: $\underline{r}(t) = \begin{pmatrix} R\cos t \\ R\sin t \\ 0 \end{pmatrix}$; $t \in [0, 2\pi]$

$$I = \oint_{C=\partial S} \underline{G}(\underline{r}) \cdot d\underline{r} = \int_0^{2\pi} dt \begin{pmatrix} -R\sin t \\ R\cos t \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -R\sin t \\ R\cos t \\ 0 \end{pmatrix}$$

$$= \int_0^{2\pi} dt\, R^2 = 2\pi R^2$$

Stokes's theorem:

$$I = \oint_{C=\partial S} \underline{G}(\underline{r}) \cdot d\underline{r} = \int_S (\nabla \times \underline{G}) \cdot d\underline{S}$$

in spherical coordinates:

$$d\underline{S} = R^2 \sin\theta \, d\theta \, d\phi \, \hat{\underline{e}}_r \qquad \phi \in [0, 2\pi], \; \theta \in [0, \pi/2]$$

$$\nabla \times \underline{G} = \begin{pmatrix} \partial_y G_z - \partial_z G_y \\ \partial_z G_x - \partial_x G_z \\ \partial_x G_y - \partial_y G_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = 2\hat{\underline{e}}_z$$

$$\Rightarrow I = 2R^2 \int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \, \sin\theta \, \underbrace{\hat{\underline{e}}_z \cdot \hat{\underline{e}}_r}_{=\cos\theta} = 4\pi R^2 \left[\tfrac{1}{2} \sin^2\theta \right]_0^{\pi/2}$$

$$= 2\pi R^2$$