Threshold modelling of spatially-dependent non-stationary extremes

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This is joint work with Philip Jonathan

Outline

- Wave height data $\rightarrow$ design of safe marine structures.
- Spatial non-stationarity and dependence
- Thresholds for non-stationary extremes
- Model parameterisation
- Theoretical and simulation studies
- Wave height data

Wave heights from the Gulf of Mexico

- Hindcasts of Y storm peak significant wave height (in metres) in the Gulf of Mexico.
  - **wave height**: trough to the crest of the wave.
  - **significant wave height**: the average of the largest 1/3 wave heights. A measure of sea surface roughness.
  - **storm peak:** largest value from each storm (hurricane): declustering.
- a 6×12 grid of 72 sites ($\approx$ 14 km apart).
- Sep 1900 to Sep 2005 : 315 storms in total.
- average of 3 observations (storms) per year, at each site.

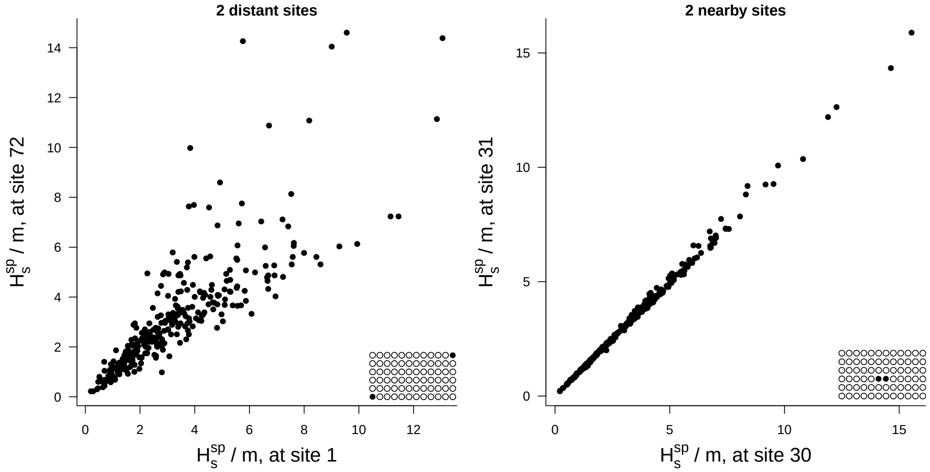
Hurricane Katrina : Aug 2005

Hurricane damage

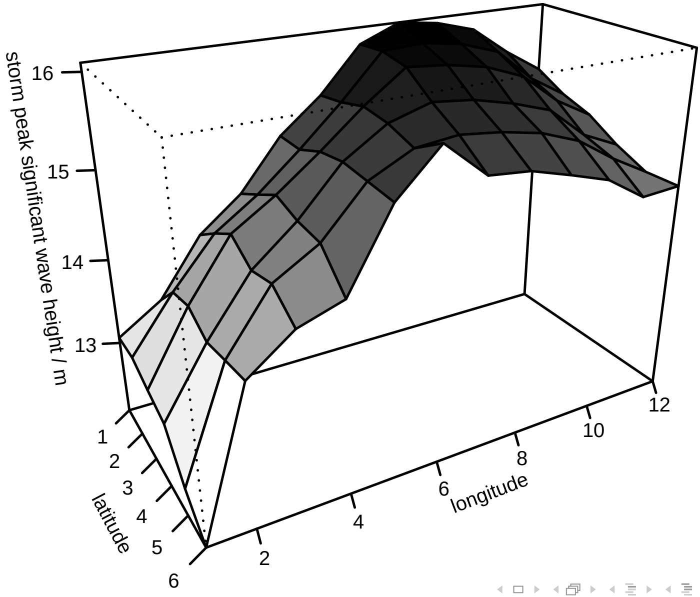
Hurricane damage

- Marginal EV regression modelling of Y, adjusting for **spatial dependence.**
- Spatial dependence not modelled explicitly: effect on marine structure is at one location.
- Estimate marginal extreme quantiles.

Spatial dependence



Spatial non-stationarity



Modelling approach

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- **Spatial non-stationarity**: model spatial effects on EV parameters as Legendre polynomials in longitude and latitude.
- **Threshold**: Use **quantile regression** to achieve approx. constant probability p of threshold exceedance over space:

  Model $100(1-p)\%$ quantile as a function of covariates.
- **Spatial dependence**.
  - Estimate parameters assuming conditional independence of responses given covariate values.
  - Adjust standard errors etc. for spatial dependence.

Extreme value regression model

Conditional on covariates $\boldsymbol{x}_{ij}$ exceedances over a high threshold $u(\boldsymbol{x}_{ij})$ follow a **2-dimensional non-homogeneous Poisson process.**

If responses $Y_{ij}, i = 1, \ldots, 72$ (space), $j = 1, \ldots, 315$ (storm) are conditionally independent:

$$L(\theta) = \prod_{j=1}^{315} \prod_{i=1}^{72} \exp\left\{ -\frac{1}{\lambda} \left[1 + \xi(\boldsymbol{x}_{ij}) \left(\frac{u(\boldsymbol{x}_{ij}) - \mu(\boldsymbol{x}_{ij})}{\sigma(\boldsymbol{x}_{ij})} \right) \right]_+^{-1/\xi(\boldsymbol{x}_{ij})} \right\}$$
$$\times \prod_{j=1}^{315} \prod_{i: y_{ij} > u(\boldsymbol{x}_{ij})} \frac{1}{\sigma(\boldsymbol{x}_{ij})} \left[1 + \xi(\boldsymbol{x}_{ij}) \left(\frac{y_{ij} - \mu(\boldsymbol{x}_{ij})}{\sigma(\boldsymbol{x}_{ij})} \right) \right]_+^{-1/\xi(\boldsymbol{x}_{ij})-1}.$$

λ : mean number of observations per year;
$\mu(\boldsymbol{x}_{ij}), \sigma(\boldsymbol{x}_{ij}), \xi(\boldsymbol{x}_{ij})$: GEV parameters of annual maxima at $\boldsymbol{x}_{ij}$;
θ : vector of all model parameters:

Covariate-dependent thresholds

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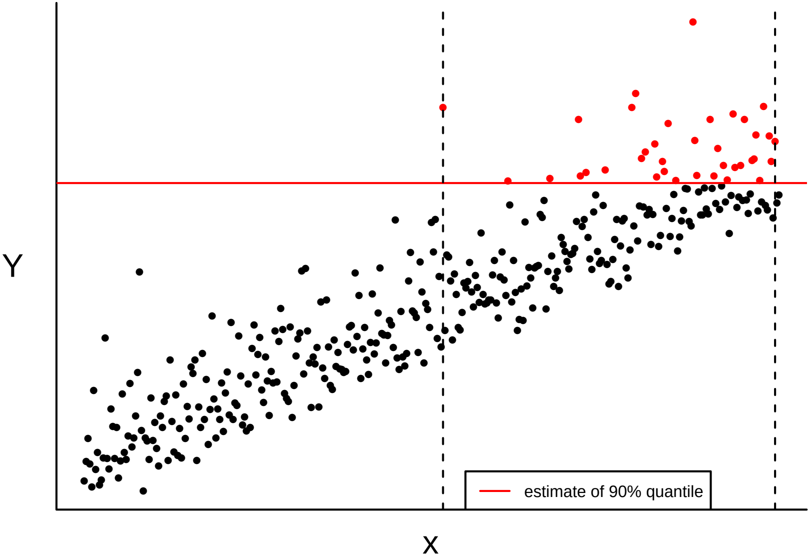
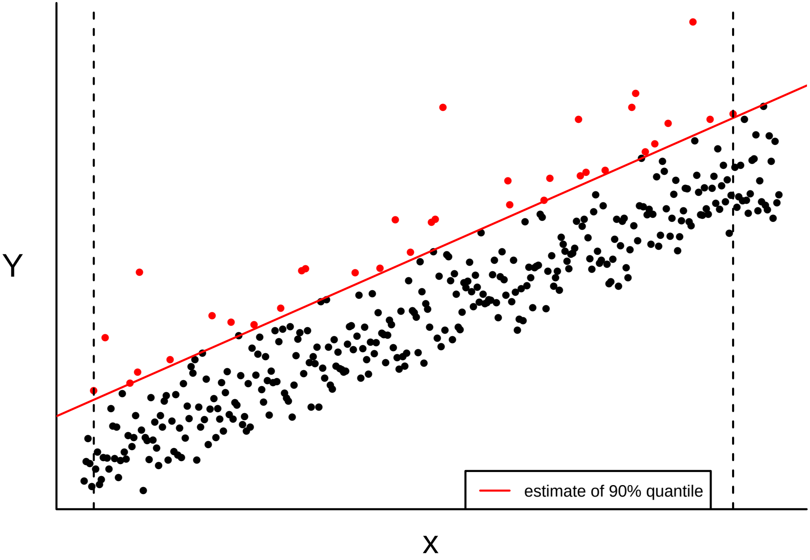
- Set $u(\boldsymbol{x}_{ij})$ by trial-and-error or by discretising $\boldsymbol{x}_{ij}$, e.g. different threshold for different locations, months etc.
- **Quantile regression (QR)** : model quantiles of a response Y as a function of covariates.

Constant threshold



Quantile regression



Model parameterisation

Let $p(\boldsymbol{x}_{ij}) = P(Y_{ij} > u(\boldsymbol{x}_{ij}))$. Then, if $\xi(\boldsymbol{x}_{ij}) = \xi$ is constant,

$$p(\boldsymbol{x}_{ij}) \approx \frac{1}{\lambda}\left[1 + \xi\left(\frac{u(\boldsymbol{x}_{ij}) - \mu(\boldsymbol{x}_{ij})}{\sigma(\boldsymbol{x}_{ij})}\right)\right]^{-1/\xi}.$$

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The form of $u(\boldsymbol{x}_{ij})$ is determined by the extreme value model:

- if $\mu(\boldsymbol{x}_{ij})$ and/or $\sigma(\boldsymbol{x}_{ij})$ are linear in $\boldsymbol{x}_{ij}$: linear QR;
- if $\log(\mu(\boldsymbol{x}_{ij})$ and/or $\log(\sigma(\boldsymbol{x}_{ij})$ is linear in $\boldsymbol{x}_{ij}$: non-linear QR.

Theoretical study (with Nicolas Attalides)

Data-generating process: for covariate values $x_1, \ldots, x_n$

$$Y_i \mid X = x_i \overset{\text{indep}}{\sim} GEV(\mu_0 + \mu_1 x_i, \sigma, \xi).$$

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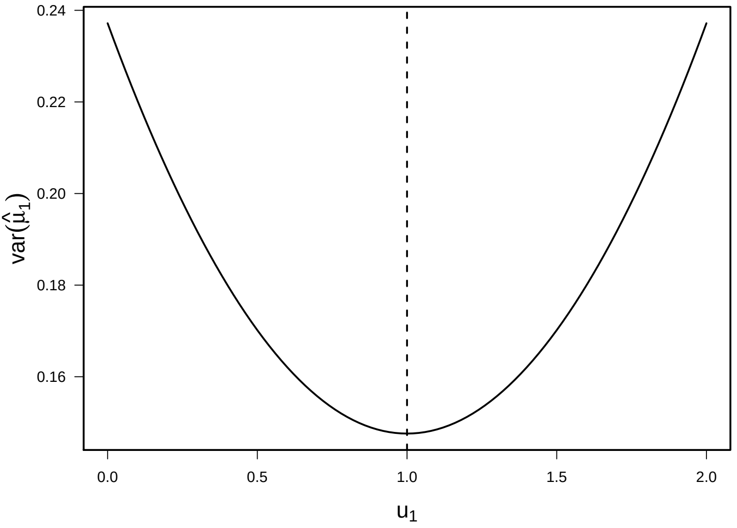
- Calculate Fisher expected information for $(\mu_0, \mu_1, \sigma, \xi)$.
- Invert to find asymptotic V-C of MLEs $\widehat{\mu}_0, \widehat{\mu}_1, \widehat{\sigma}, \widehat{\xi}$ and hence $\text{var}(\widehat{\mu}_1)$.
- Find the value of u_1 that minimises $\text{var}(\widehat{\mu}_1)$.

Preliminary findings

Let $\tilde{u}_1$ be the value of u_1 that minimises $\mathrm{var}(\widehat{\mu}_1)$.

- If covariate values $x_1, \ldots, x_n$ are symmetrically distributed then $\tilde{u}_1 = \mu_1$ (quantile regression).

$\mu_1 = 1$: symmetric x



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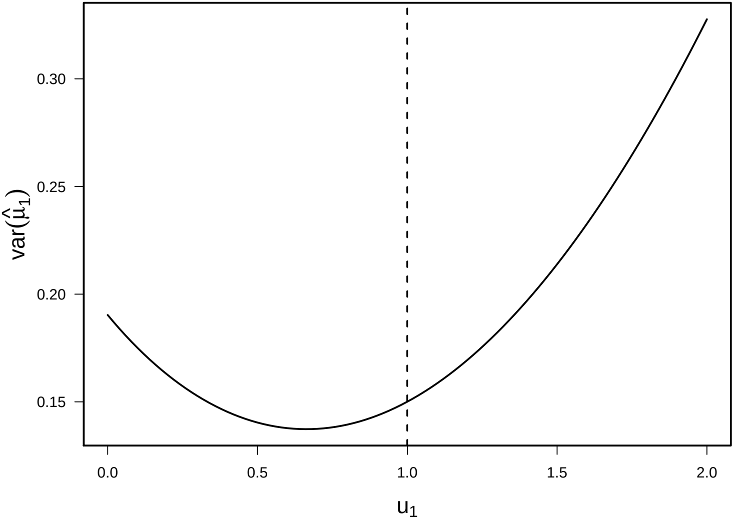
- If $x_1, \ldots, x_n$ are positive (negative) skew then $\tilde{u}_1 < \mu_1$ ($\tilde{u}_1 > \mu_1$).

$\mu_1 = 1$: positive skew x (skewness = 1)



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Extensions:

- More general models.
- Effect of model mis-specification due to low threshold;

Adjustment for spatial dependence

Independence log-likelihood:

$$l_{IND}(\theta) = \underset{\text{(storms)}}{\sum_{j=1}^{k}} \; \underset{\text{(space)}}{\sum_{i=1}^{72}} \log f_{ij}(y_{ij}; \theta) = \sum_{j=1}^{k} l_j(\theta).$$

In regular problems, as $k \to \infty$,

$$\widehat{\theta} \to N(\theta_0, H^{-1}\, V\, H^{-1}),$$

- H = expected Hessian: $\mathrm{E}\left(\frac{\partial^2}{\partial\theta^2}\, l_{IND}(\theta_0)\right)$;
- $V = \mathrm{var}\left(\frac{\partial}{\partial\theta}\, l_{IND}(\theta)\right)$

Adjustment of $l_{IND}(\theta)$

Estimate

- H by observed Hessian, at $\widehat{\theta}$;
- V by $\sum_{j=1}^{k} U_j\left(\widehat{\theta}\right)^T U_j\left(\widehat{\theta}\right)$, $\qquad U_j(\theta) = \dfrac{\partial l_j(\theta)}{\partial \theta}$.

Let $\widehat{H}_A = \left(-\widehat{H}^{-1}\,\widehat{V}\,\widehat{H}^{-1}\right)^{-1}$.

Chandler and Bate (2007):

$$l_{ADJ}(\theta) = l_{IND}(\widehat{\theta}) + \frac{(\theta - \widehat{\theta})'\,\widehat{H}_A\,(\theta - \widehat{\theta})}{(\theta - \widehat{\theta})'\,\widehat{H}\,(\theta - \widehat{\theta})}\left(l_{IND}(\theta) - l_{IND}(\widehat{\theta})\right),$$

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- Adjust $l_{IND}(\theta)$ so that its Hessian is H_A at $\widehat{\theta}$ rather than H.
- This adjustment preserves the usual asymptotic distribution of the likelihood ratio statistic.

Findings of a simulation study

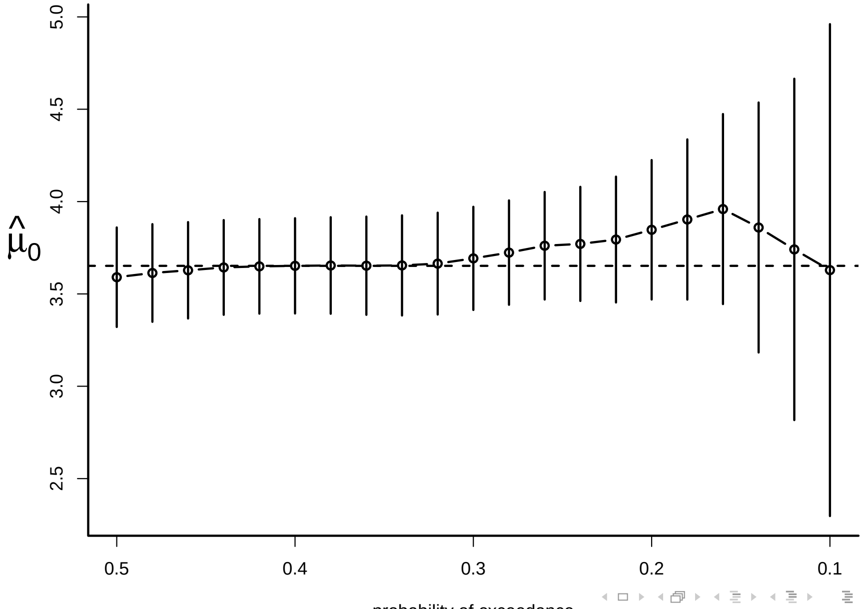
- Data simulated with spatial and temporal dependence and with spatial variation.
- Slight underestimation of standard errors : uncertainty in threshold ignored.
- Uncertainties in covariate effects of threshold are negligible compared to the uncertainty in the level of the threshold.
- Estimates of regression effects from QR and EV models are very close : both estimate extreme quantiles from the same data.
- To a large extent fitting the EV model accounts for uncertainty in the covariate effects at the level of the threshold.

Summary of wave height modelling

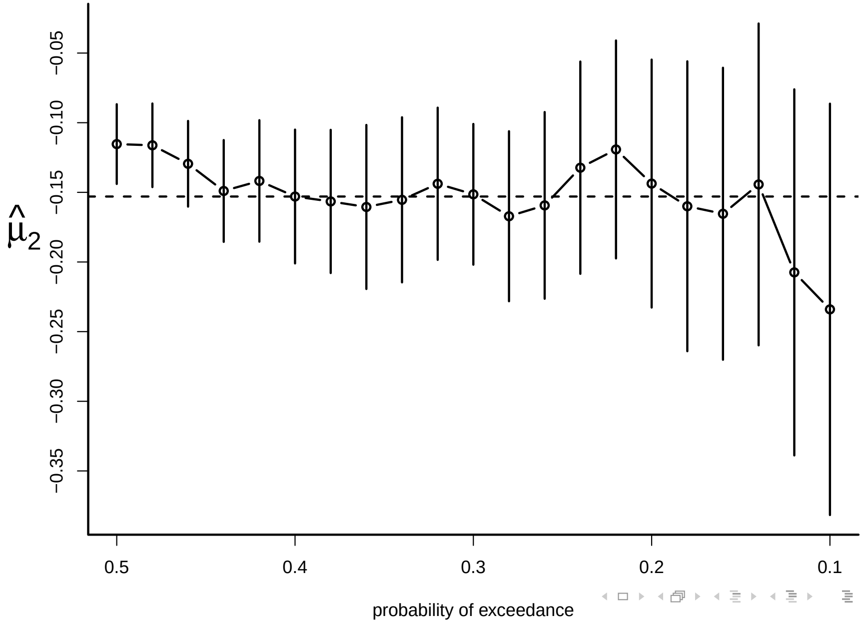
Threshold selection:

- Iterative: form of threshold depends on model.
- For given EV model set threshold using appropriate QR model.
- Choice of exceedance probability p: look for stability in parameter estimates.
- Based on μ (and u) quadratic in longtiude and latitude, σ and ξ constant ...

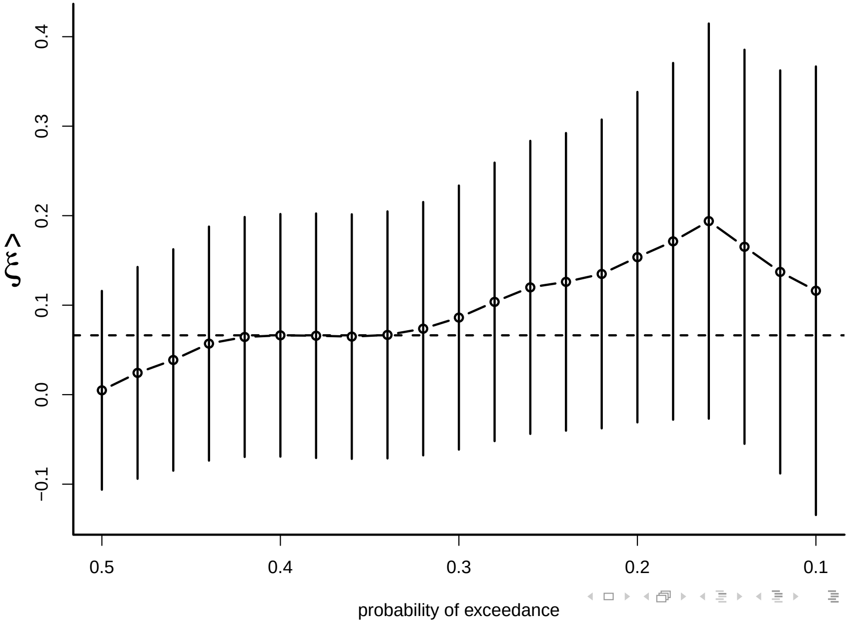
Threshold selection : μ intercept



Threshold selection : μ coeff of latitude



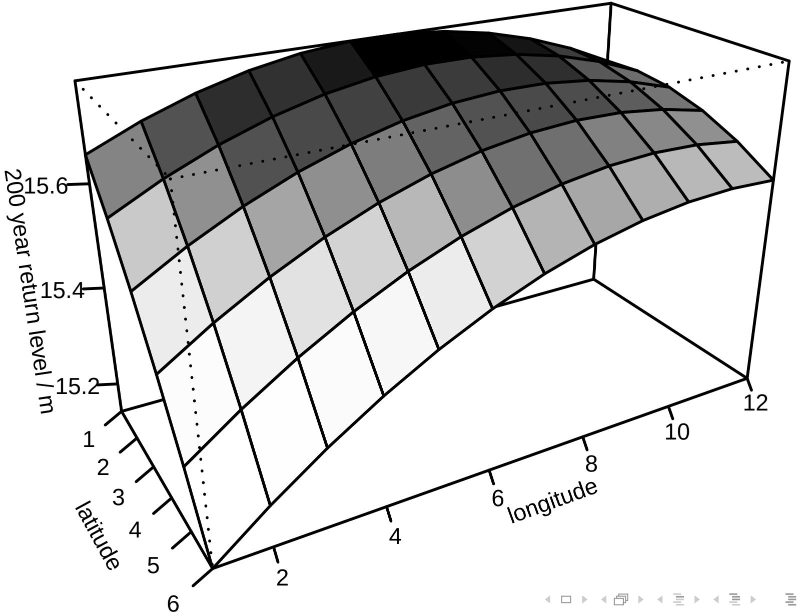
Threshold selection : ξ



Summary of results

- Choice of p: look for stability in parameter estimates. Use $p = 0.4$.
- Model diagnostics : slight underestimation at very high levels, but consistent with estimated sampling variability.
- QR model and EV model agree closely.
- $\widehat{\xi} = 0.066$, with 95% confidence interval $(-0.052, 0.223)$.
- Estimated 200 year return level at (long=7, lat=1) is 15.78m with 95% confidence interval $(12.90, 22.28)$m.

Conditional 200 year return levels



Discussion

Quantile regression:

- a simple and effective strategy to set thresholds for non-stationary EV models;
- supported by simulation study;
- theoretical work is on-going;

Kyselý, J., et al. (2010): QR to set time-dependent thresholds.

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Comments

- Simple approach: relatively accessible to engineers.
- Simpler r-largest order statistic analysis preferable for these data - but not for irregularly-spaced covariates.
- $(\widehat{U}\,\widehat{H}^{-1})^T\widehat{U}\,\widehat{H}^{-1}$ more comp. stable than $\widehat{H}^{-1}\widehat{V}\,\widehat{H}^{-1}$.
- Wave direction; individual hurricane locations.

References

Chandler, R. E. and Bate, S. B. (2007) Inference for clustered data using the independence loglikelihood. *Biometrika* **94 (1)**, 167–183.

Kyselý, J., Picek, J. and Beranová, R. (2010) Estimating extremes in climate change simulations using the peaks-over-threshold method with a non-stationary threshold *Global and Planetary Change*, **72**, 55-68.

Northrop, P. J. and Jonathan, P. Threshold modelling of spatially-dependent non-stationary extremes with application to hurricane-induced wave heights. Published online in *Environmetrics*. **Discussion and rejoinder to follow shortly.**

Thank you for your attention.