Non-homogeneous Random Walks on a Half Strip

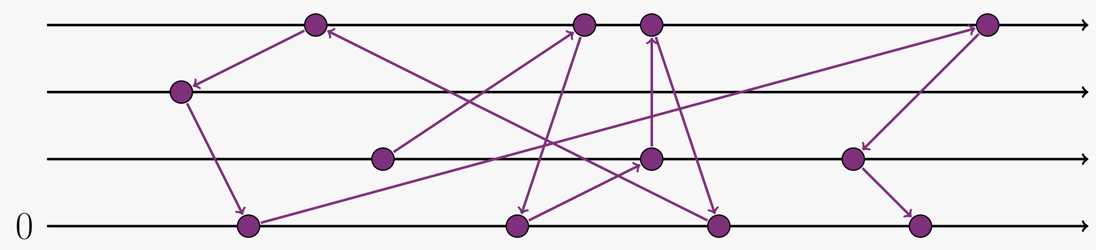
Chak Hei Lo

Department of Mathematical Sciences, University of Durham, UK.

c.h.lo@durham.ac.uk

Joint work with

Andrew R. Wade



Random walk on a half strip

Motivation

- Many stochastic processes arising in applications exhibit a range of possible behaviours depending the values of certain key parameters.
- Investigating phase transitions for such systems leads to interesting and challenging mathematics.
- We aim to extend known criteria for classifying recurrence and transience in a particular near-critical Markov model.
- This will serve as a prototypical model for developing novel aspects of the semi-martingale method, which can then be used in applications.

Markov chain on a strip model

- Let S be a finite non-empty set, and let Σ be a subset of $\mathbb{R}_+ \times S$ that is *locally finite*, i.e., $\Sigma \cap ([0, r] \times S)$ is finite for all $r \in \mathbb{R}_+$.
- We will study the asymptotic behaviour of time-homogeneous irreducible Markov chain (X_n, η_n), $n \in \mathbb{Z}_+$, on Σ.
- The local finiteness assumption ensures that transience of the Markov chain (X_n, η_n) is equivalent to $\lim_{n\to\infty} X_n = +\infty$, a.s.
- Note that neither of the coordinates is necessarily Markov.

Applications

Some example applications with a number of literature are the following.

- Queueing theory: modulated queues [5].
- Mathematical Finance: regime-switching processes.
- Physics: physical processes with internal degrees of freedom, in the form of correlated random walk [3].

Notation and assumptions

Now we need the following assumptions to proceed.

- Assume the displacement of the X-coordinate has bounded p-moments for some $p < \infty$.
  - $(\mathbf{B_p})$ There exists a constant $C_p < \infty$ such that
  $$\mathbb{E}[|X_{n+1} - X_n|^p \mid X_n = x, \eta_n = i] \le C_p \ a.s. \ \forall n.$$
- Define $q_{ij}(x) = \mathbb{P}[\eta_{n+1} = j \mid (X_n, \eta_n) = (x, i)]$ and assume
  - $(\mathbf{Q_\infty})$ $\lim_{x\to\infty} q_{ij}(x) = q_{ij}$ exists for all $i, j \in S$, and (q_{ij}) is an irreducible stochastic matrix.
- Let $\boldsymbol{\pi}$ be the unique stationary distribution on S corresponding to (q_{ij}).
- Naturally, we want to specify the movement of the chain by its first and second moments in the $\mathbb{R}_+$-coordinates.
$$\mu_i(x) := \mathbb{E}\left[X_{n+1} - X_n \mid X_n = x, \eta_n = i\right].$$
$$\sigma_i(x) := \mathbb{E}\left[(X_{n+1} - X_n)^2 \mid X_n = x, \eta_n = i\right].$$

Notice that $\mu_i(x)$ and $\sigma_i(x)$ are finite if $(\mathbf{B_p})$ holds for some $p \ge 1$ and some $p \ge 2$ respectively.

Lamperti drift

Lamperti drift classification

- If there is a constant drift, i.e. $\mu_i(x) = d_i + o(1)$, then the recurrence classification depends on the sign of $\sum_{i\in S} \pi_i d_i$. Positive total average drift leads to transient and negative leads to positive recurrence [1, 2]. The remaining case when we have zero average is the critical case.
- One natural guess would just be null-recurrence whenever the condition is satisfied but this is not always true. The case here is very subtle and it depends on a lot more than just the first term of the drift. We should assume the following condition.
  - $(\mathbf{D_L})$ There exist $c_i \in \mathbb{R}$ and $s_i^2 \in \mathbb{R}_+$, with at least one s_i^2 non-zero, such that for all $i \in S$, as $x \to \infty$, $\mu_i(x) = \frac{c_i}{x} + o(x^{-1})$ and $\sigma_i(x) = s_i^2 + o(1)$.
- The case when S is a singleton is the well-known Lamperti Problem [4].

Theorem 1. [2] *Suppose that $(\mathbf{B_p})$ holds for some $p > 2$, and conditions $(\mathbf{Q_\infty})$ and $(\mathbf{D_L})$ hold. Then the following classification applies.*

- *If $\sum_{i\in S}(2c_i - s_i^2)\pi_i > 0$, then X_n is transient.*
- *If $|\sum_{i\in S} 2c_i\pi_i| < \sum_{i\in S} s_i^2\pi_i$, then X_n is null-recurrent.*
- *If $\sum_{i\in S}(2c_i + s_i^2)\pi_i < 0$, then X_n is positive-recurrent.*

[With slightly better error bounds in $(\mathbf{Q_\infty})$ and $(\mathbf{M_L})$ we can show that the boundary cases are null-recurrent.]

Moments of Lamperti drift type

The degree of recurrence can be quantified by investigating existence of moments of the return times $\tau_x := \min\{n \ge 0 : X_n \le x\}$. More moments exists means the process is more recurrence in asymptotical sense. Here is the necessary (and sufficient with Theorem 3) condition for the existence of moments.

Theorem 2 (L., Wade, 2015)**.** *Suppose that $(\mathbf{B_p})$ holds for some $p > 2$, and conditions $(\mathbf{Q_\infty})$ and $(\mathbf{D_L})$ hold. If*
$$\sum_{i\in S}[2c_i + (2\theta - 1)s_i^2]\pi_i < 0,$$
then for any $s \in \left[0, \theta \wedge \frac{p}{2}\right]$, we have $\mathbb{E}[\tau_x^s] < \infty$.

The proof is base on the idea of Lyapunov functions. Using a different starting function with the same technique, we can show the other side of the story as the following.

Theorem 3 (L., Wade, 2015)**.** *Suppose that $(\mathbf{B_p})$ holds for some $p > 2$, and conditions $(\mathbf{Q_\infty})$ and $(\mathbf{D_L})$ hold. If*
$$\sum_{i\in S}[2c_i + (2\theta - 1)s_i^2]\pi_i > 0,$$
for some $\theta > 0$, then for any $s \in \left[\theta, \frac{p}{2}\right]$, we have $\mathbb{E}[\tau_x^s] = \infty$.

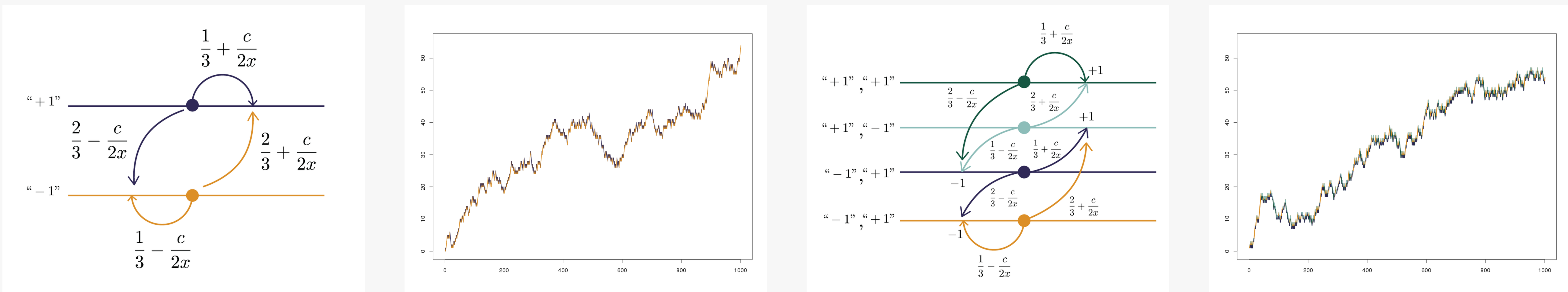


Figure 1: Two simulations of 10^3 steps of correlated random walks, as an application of the half strip model. The first and third figures show how the walk moves, with one-step and two-steps correlated respectively. The second and fourth figures display the displacement in the X_n direction against the number of steps.

Generalized Lamperti drift

Assumptions

- We define $\mu_{ij}(x)$ to represent the average drift at x from line i to j, i.e.
$$\mu_{ij}(x) = \mathbb{E}_{x,i}[(X_{n+1} - X_n)\mathbf{1}\{\eta_{n+1} = j\}].$$
This alerts us that the interaction between the lines is actually crucial in this case. We define the generalized Lamperti drift as follows.
  - $(\mathbf{D_G})$ For $i, j \in S$ there exist $d_i \in \mathbb{R}$, $e_i \in \mathbb{R}$, $d_{ij} \in \mathbb{R}$ and $t_i^2 \in \mathbb{R}_+$, with at least one t_i^2 non-zero, such that
    (a) for all $i \in S$, $\mu_i(x) = d_i + \frac{e_i}{x} + o(x^{-1})$ as $x \to \infty$;
    (b) for all $i \in S$, $\sigma_i^2(x) = t_i^2 + o(1)$ as $x \to \infty$;
    (c) for all $i, j \in S$, $\mu_{ij}(x) = d_{ij} + o(1)$ as $x \to \infty$; and
    (d) $\sum_{i\in S} \pi_i d_i = 0$.
- After settling the control of the moments, we also need some extra condition on the transitional probability to precisely pinpoint the phase transition. Here is the assumption.
  - $(\mathbf{Q_G})$ There exist $\gamma_{ij} \in \mathbb{R}$, such that $q_{ij}(x) = q_{ij} + \frac{\gamma_{ij}}{x} + o(x^{-1})$, where (q_{ij}) is a stochastic matrix.

Generalized Lamperti drift classification

Now we give our main recurrence classification for the model with generalized Lamperti drift. Notice that although (a_i) are not unique, but nevertheless the expression in which they appear in the following theorem are invariant under translation of the (a_i), and so the criteria are well-defined.

Theorem 4 (L., Wade, 2015)**.** *Suppose that $(\mathbf{B_p})$ holds for some $p > 2$, and conditions $(\mathbf{Q_G})$ and $(\mathbf{D_G})$ hold. Define a_i to be the unique solution up to translation of the system of equations $d_i + \sum_{j\in S}(a_j - a_i)q_{ij} = 0 \ \forall\, i \in S$. Then the following sufficient conditions apply.*

- *If $\sum_{i\in S}[2e_i - t_i^2 + 2\sum_{j\in S} a_j(\gamma_{ij} - d_{ij})]\pi_i > 0$ then X_n is transient.*
- *If $|\sum_{i\in S}(2e_i + 2\sum_{j\in S} a_j\gamma_{ij})\pi_i| < \sum_{i\in S}(t_i^2 + 2\sum_{j\in S} a_j d_{ij})\pi_i$ then X_n is null-recurrent.*
- *If $\sum_{i\in S}[2e_i + t_i^2 + 2\sum_{j\in S} a_j(\gamma_{ij} + d_{ij})]\pi_i < 0$ then X_n is positive-recurrent.*

[With slightly better error bounds in $(\mathbf{Q_G})$ and $(\mathbf{D_G})$ we can show that the boundary cases are null-recurrent.]

Moments of Generalized Lamperti drift type

We also have similar criteria for the existence and non-existence of moments of generalized Lamperti drift type.

Theorem 5 (L., Wade, 2015)**.** *Suppose that $(\mathbf{B_p})$ holds for some $p > 2$, and conditions $(\mathbf{Q_G})$ and $(\mathbf{D_G})$ hold. Define a_i to be the unique solution up to translation of the system of equations $d_i + \sum_{j\in S}(a_j - a_i)q_{ij} = 0 \ \forall\, i \in S$. If*
$$\sum_{i\in S}[2e_i + (2\theta - 1)t_i^2 + 2\sum_{j\in S} a_j(\gamma_{ij} + (2\theta - 1)d_{ij})]\pi_i < 0,$$
then for any $s \in \left[0, \theta \wedge \frac{p}{2}\right]$, we have $\mathbb{E}[\tau_x^s] < \infty$.

Theorem 6 (L., Wade, 2015)**.** *Suppose that $(\mathbf{B_p})$ holds for some $p > 2$, and conditions $(\mathbf{Q_G})$ and $(\mathbf{D_G})$ hold. Define a_i to be the unique solution up to translation of the system of equations $d_i + \sum_{j\in S}(a_j - a_i)q_{ij} = 0 \ \forall\, i \in S$. If*
$$\sum_{i\in S}[2e_i + (2\theta - 1)t_i^2 + 2\sum_{j\in S} a_j(\gamma_{ij} + (2\theta - 1)d_{ij})]\pi_i > 0,$$
then for any $s \in \left[\theta, \frac{p}{2}\right]$, we have $\mathbb{E}[\tau_x^s] = \infty$.

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