A wavelet filter enhancement scheme with a fast integral B-wavelet transform and pyramidal multi-B-wavelet algorithm

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Abstract

A construction paradigm is proposed to refine a class of wavelet bases such that the filter characteristics are enhanced. In particular, it is shown that the entire Mth order B-wavelet family belong to this class. Exploiting some recent work, a fast integral wavelet transform is found for the refined B-wavelet family. Moreover, this refined basis can be placed into the setting of a multiresolution analysis that has a multiplicity greater than one. Accordingly, a fast, discrete, pyramidal algorithm is realised.

Key words: Multiwavelet, wavelet modification, fast continuous wavelet transform, lifting scheme, filter characteristic

1 Introduction

With the advent of the lifting scheme came a general approach to modify wavelets such that a selection of their properties could be systematically altered. We propose a disparate, specialised approach to mitigate the side lobes of a wavelet filter characteristic whilst preserving differentiability, symmetry, anti-symmetry, compact support, and the existence of an analytical expression.

Devised by Sweldens [12], the lifting scheme can be used to emend an existing wavelet and scaling function basis to produce so called second generation wavelets. Such wavelets are not necessarily formed by dilations and translations of a single scalar function. A distinctive feature of the scheme is that the

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wavelet construction uses a linear combination of the initial scaling functions at the same scale level as the wavelet, whereas an orthodox wavelet construction uses scaling functions at the next finest level. To paraphrase Sweldens [13], the lifting scheme uses the 'aunt' scaling functions rather than the 'sister' scaling functions. Aside from giving rise to a new class of wavelets, the lifting scheme offers a custom based design as well as a means to contrive algorithms that outperform the classical fast wavelet transform.

Although our modification scheme can give rise to second generation wavelets, it differs from the lifting scheme because a linear combination of integer dilated and translated 'sister' scaling functions are employed as the building blocks of the wavelet. The principal aim of the construction is to minimise the side lobe to main lobe power and magnitude ratios of the filter characteristic about carefully chosen sub-intervals such that differentiability, symmetry, anti-symmetry, compact support, and the existence of an analytical expression are all preserved.

Section 2 describes some measures that assess the efficacy of wavelet filter characteristics. We generalise these measures and, in Section 3, some numerical results illustrate that our modification scheme achieves enhanced characteristics for the B-wavelet family. Adapting recent work by by Muñoz et al. [11] and Unser et al. [14], Section 4 shows how a fast integral wavelet transform can be formulated for the modified B-wavelets. First realised by Alpert [1] and Geronimo et al. [6], the multi-scaling function is discussed in Section 5. Finally, we consider the modified B-wavelet construction in the setting of a multiresolution analysis generated by a multi-scaling function.

2 Measures for the frequency characteristic of a wavelet

For $\alpha, \tau \in \mathbb{R}$, the integral wavelet transform of a function $f \in L_2(\mathbb{R})$ takes the form

$$(\mathcal{W}_\psi f)(\alpha, \tau) = |\alpha|^{-1/2} \int_{\mathbb{R}} f(x) \overline{\psi\left(\frac{\tau - x}{\alpha}\right)} dx. \tag{1}$$

That is $(\mathcal{W}_\psi f)(\alpha, \tau) = (f^\wedge \Psi_\alpha^\wedge)^\vee(\tau)$, with $\Psi_\alpha := \sqrt{2\pi}\,|\alpha|^{-1/2}\,\psi(\alpha^{-1}\cdot)$, and where $^\wedge$ and $^\vee$ denote the Fourier, and inverse Fourier, transform operations. Accordingly, $\Psi_\alpha^\wedge$ can be thought of as a filter transfer function with a normalised filter characteristic $|\psi^\wedge|$. Invertibility of the integral wavelet transform requires that the wavelet satisfies the admissability condition, namely

$$\int_{\mathbb{R}} |\psi^\wedge(\xi)|^2 \, \frac{d\xi}{|\xi|} < \infty.$$

Since it follows that $\psi^\wedge(0) = 0$, wavelets are often thought of in the context of band pass filters. A critical feature of the wavelet transform is that the

width of the time or space window narrows when the wavelet is assessing the high frequency content of f, and it expands when assessing the lower frequencies. One is therefore left with the problem of constructing a function that is, in some sense, well localised in both time and frequency. However, except from the trivial example of zero, functions cannot have compact support in both time and frequency. Consequently, a more meaningful measure of time-frequency localisation is employed, as follows.

Definition 2.1 *Suppose* $h(x), |x|^{1/2} h(x), xh(x) \in L_2(X)$. *Then the centre of* h *on* X *is defined as*

$$x_X^*(h) := \frac{\left\| |x|^{1/2} h(x) \right\|_{L_2(X)}^2}{\|h(x)\|_{L_2(X)}^2}$$

and the radius of h *is defined as*

$$\Delta_X(h) := \frac{\|(x - x_X^*(h))h(x)\|_{L_2(X)}}{\|h(x)\|_{L_2(X)}}.$$

The uncertainty of a wavelet, $U(\psi) := \Delta_{\mathbb{R}}(\psi)\Delta_{\mathbb{R}^+}(\psi^\wedge)$ is proportional to the area of the time-frequency window and is bounded from below by the well known Heisenberg-Weyl uncertainty condition $U(\psi) \geq 1/2$. The uncertainty measure gives a broad indication of how well localised a wavelet is. More specific measures exist that are used to appraise the filter characteristic of a wavelet. Two such methods of measurement, discussed below, recognise that, in order to perform a band pass operation efficiently, the power in the main lobe must be significantly greater than the side lobes. Such an observation is not lost on Dovis et al. [5], who discuss the virtues of low side lobes in the context of digital communications. In the following we denote those zeros of $\psi^\wedge(\xi)$ that lie on the positive half of the frequency axis as $0 < \xi_0 < \xi_1 < \ldots$. The first measure takes the ratio of the power of all the side lobes and divides it by the power of the main lobe [2].

Definition 2.2 *The side lobe/main lobe power ratio of the filter characteristic* $|\psi^\wedge|$ *is defined as*

$$\mathcal{M}\psi := \frac{\|\psi^\wedge\|_{L_2(\xi_0,\infty)}^2}{\|\psi^\wedge\|_{L_2(0,\xi_0)}^2}.$$

The second measure compares the maximum taken at the main lobe and the maximum at the neighbouring lobe. Indeed, as Chui, [2] points out, this comparison is favoured in many engineering applications such as antenna design.

Definition 2.3 *The maximal side lobe/main lobe ratio of the filter character-*

istic $|\psi^\wedge|$ *is defined as*

$$\mathcal{M}^*\psi := \frac{\|\psi^\wedge\|_{L_\infty(\xi_0,\xi_1)}}{\|\psi^\wedge\|_{L_\infty(0,\xi_0)}}.$$

We propose a generalisation of both measures. The ratio in Definition 2.2 can be localised by taking the side lobe power over a finite interval Ω instead of the interval (ξ_0, ∞).

Definition 2.4 *The localised side lobe/main lobe power ratio of the filter characteristic* $|\psi^\wedge|$ *is defined as*

$$\mathcal{M}_\Omega\psi := \frac{\|\psi^\wedge\|^2_{L_2(\Omega)}}{\|\psi^\wedge\|^2_{L_2(0,\xi_0)}}$$

for some interval $\Omega \subseteq \mathbb{R}^+_* := \mathbb{R}^+ \setminus \{0\}$.

In particular, this measure can be calculated over the first N side lobes by choosing $\Omega = (\xi_0, \xi_N)$. It can be seen immediately that Definition 2.2 is a specific case of $\mathcal{M}_\Omega$, with $\Omega = (\xi_0, \infty)$. Alternatively, the ratio in Definition 2.3 can be extended by replacing the numerator with an average of the first N side lobe maxima.

Definition 2.5 *Let* $\Omega_n = (\xi_n, \xi_{n+1})$, *and* $\xi_{-1} = 0$. *The globalised maxima side lobe/main lobe ratio of the filter characteristic* $|\psi^\wedge|$ *is defined as*

$$\mathcal{M}^*_N\psi := N \|\psi^\wedge\|^{-1}_{L_\infty(\Omega_{-1})} \sum_{n=0}^{N-1} \|\psi^\wedge\|_{L_\infty(\Omega_n)}.$$

When $N = 1$, this reduces to Definition 2.3. To construct wavelets which perform efficiently with respect to such measures, it is clearly necessary to concentrate upon reducing the side lobe power in some neighbourhood about the main lobe.

3 Modifying wavelet bases to optimise frequency localisation

The following method is proposed as a means of altering an existing wavelet basis such that the side lobe/main lobe measures of a wavelet are enhanced. The technique modifies a wavelet by adding dilated versions of the wavelet to itself. These linear combinations are constructed such that the power associated with the side lobes of the filter characteristic is attenuated. After a general construction is introduced, the application to the B-wavelet family is discussed.

A wavelet ψ is considered with $\operatorname{supp}\psi \subseteq [0, X)$. The following construction will make use of dilated, periodically extended versions of ψ onto the interval $[0, X)$. Let the set of integers $\{d_j\}$, with $d_1 = 1$, be defined by

$$d_{j+1} = \min_{n \in 2\mathbb{N}-1} \left\{ n > d_j \text{ and } \max_{n^\alpha | d} \alpha = 1 \right\},$$

for all $j = 1, \ldots, J$, and

$$d := \prod_{j=1}^{J} d_j,$$

and where $a \mid b$ denotes the condition that a divides b. We define the operators $\mathcal{T}_j : [0, X) \mapsto [0, X)$ by

$$\mathcal{T}_j \psi := \sum_{n=0}^{X(d_j - 1)} \psi(d_j \cdot - n).$$

The modified wavelet, denoted by ψ^J, is defined in terms of the operators $\mathcal{T}_j$ as follows:

$$\psi^J := \sum_{j=1}^{J} \mu(d_j) \Gamma_j \mathcal{T}_j \psi, \tag{2}$$

where $\mu : \mathbb{N} \mapsto \{0, \pm 1\}$ is the Möbius arithmetic function, given by

$$\mu(n) := \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0, & \text{otherwise} \end{cases},$$

and the Γ_j comprise a set of suitably chosen coefficients. The Möbius function is employed here due to the utility afforded by the following result, taken from Number Theory.

Lemma 3.1 *Let μ denote the Möbius function. Then, for $n \in \mathbb{N}$,*

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \textit{for } n = 1 \\ 0, & \textit{otherwise} \end{cases}.$$

In view of our aim to attenuate the wavelet filter characteristic side lobe power, we present the following result.

Theorem 3.2 *Consider the modified wavelet ψ^J, as defined by (2) with the*

set $\{d_j\}_1^J$ *and, for any* $k, m, n \in \mathbb{Z}$*, assume that*

$$\frac{\psi^\wedge(2n\pi)}{\psi^\wedge(2nm\pi)} = \frac{\psi^\wedge(2k\pi)}{\psi^\wedge(2km\pi)}, \qquad \text{if} \quad \psi^\wedge(2nm\pi)\psi^\wedge(2km\pi) \neq 0. \tag{3}$$

Then Γ_j *can be chosen such that*

$$\psi^{J\wedge}(2n\pi) = 0, \qquad 1 < n \leq d_J.$$

Proof The filter characteristic of ψ^J is determined by

$$(\mathcal{T}_j\psi)^\wedge(\xi) = \frac{1}{d_j}\psi^\wedge\left(\frac{\xi}{d_j}\right) K_j(\xi)$$

with

$$K_j(\xi) = \sum_{r=0}^{X(d_j-1)} e^{-ir\xi/d_j} = \frac{1 - e^{-i(d_jX - X + 1)\xi/d_j}}{1 - e^{-i\xi/d_j}}.$$

Note that $K_j(2d_j\pi) = K_j(2n\pi)|_{d_j|n}$, and assumption (3) implies that

$$\frac{\psi^\wedge(2d_j\pi)}{\psi^\wedge(2\pi)} = \left.\frac{\psi^\wedge(2n\pi)}{\psi^\wedge(2nd_j^{-1}\pi)}\right|_{d_j|n}.$$

It follows that choosing Γ_j to be

$$\Gamma_j = \frac{d_j\psi^\wedge(2d_j\pi)}{K_j(2d_j\pi)\psi^\wedge(2\pi)},$$

ensures that, for $n \in d_j\mathbb{N}$, we have

$$\psi^{J\wedge}(2n\pi) = \psi^\wedge(2n\pi)\sum_{d_j|n}^{d_J} \mu(d_j).$$

Lemma 3.1 concludes the proof. ■

Therefore, given a wavelet basis ψ, this process will introduce zeros into the modified frequency characteristic $|\psi^{J\wedge}|$. Of particular relevance are the zeros at $\xi = 2n\pi$ for $1 < n \leq d_J$ which may help to reduce the power ratios introduced in Definitions 2.4 and 2.5.

It should be noted that this refinement technique does not necessarily guarantee that ψ^J can be generated from a multiresolution analysis. Furthermore, the scheme is only relevant for the class of wavelets that satisfies the condition stipulated by (3). However, since this process involves forming linear combinations of dilated versions of a wavelet, it will preserve the symmetry, anti-symmetry, differentiability, and support of the original wavelet.

The refinement method will now be applied to the family of B-wavelets. A quantitative comparison is then made between the frequency characteristics of the original wavelets and those of the modified wavelets.

We let ϕ_M and ψ_M denote the Mth order B-spline and the associated Mth order B-wavelet, constructed by Chui and Wang [3]. Let χ_Ω denote the characteristic, or indicator, function on some interval Ω. The first order B-spline is the Haar scaling function defined by $\phi_1 := \chi_{[0,1)}$ and the first order B-wavelet is the Haar wavelet $\psi_1 = \chi_{[0,1/2)} - \chi_{[1/2,1)}$. Other members of the B-spline family are generated by forming an $(M-1)$-fold convolution with ϕ_1. That is

$$\phi_M := \phi_1 * \phi_{M-1}. \tag{4}$$

The two-scale relation for ϕ_1 can be expressed in the frequency domain as

$$\phi_1^\wedge = H(z)\phi_1^\wedge\left(\frac{\cdot}{2}\right),$$

with $H(z) = (1+z)/2$, and $z = z(\xi) := e^{-i\xi/2}$. Hence, taking the Fourier transform of both sides of (4) leads to the two-scale relation for ϕ_M, namely

$$\phi_M^\wedge = \left(\frac{1+z}{2}\right)^M \phi_M^\wedge\left(\frac{\cdot}{2}\right).$$

The B-wavelets are constructed in the multiresolution analysis generated by the B-splines. In the original domain, the B-wavelets can be expressed as

$$\psi_M = \sqrt{2}\sum_k g_M(k)\phi_M(2\cdot - k), \tag{5}$$

and in the Fourier domain we have

$$\psi_M^\wedge(\xi) = \frac{1}{\sqrt{2}}\sum_k g_M(k) z^k \phi_M^\wedge\left(\frac{\xi}{2}\right) =: G_M(z)\phi_M^\wedge\left(\frac{\xi}{2}\right). \tag{6}$$

Following the treatment of Chui [2], the non-zero coefficients of $g_M(k)$ can be calculated by

$$g_M(k) = \frac{(-1)^k}{2^{M-1}\sqrt{2}}\sum_{\ell=0}^{M}\binom{M}{\ell}\phi_{2M}(k-\ell+1), \qquad k = 0,\ldots,3M-2,$$

together with the recursive scheme

$$\phi_j(n) = \frac{n}{j-1}\phi_{j-1}(n) + \frac{j-n}{j-1}\phi_{j-1}(n-1), \qquad j = 3,\ldots,2M,$$

with $\phi_2(n) = \delta_{n,1}$, for $n \in \mathbb{Z}$. Given the B-wavelet family above, we require the coefficients $\{\Gamma_j\}$ in order to apply the refinement process. Recall

$$\Gamma_j = d_j K_j^{-1} \frac{\psi^\wedge(2n\pi)}{\psi^\wedge(2n\pi/d_j)}. \tag{7}$$

By using the two-scale relation (6), the fraction involving the function ψ in (7) can be simplified by putting $k = n/d_j$. We have

$$\frac{\psi^\wedge(2n\pi)}{\psi^\wedge(2n\pi/d_j)} = \frac{G\left(e^{-id_jk\pi}\right)}{G\left(e^{-ik\pi}\right)} \frac{\phi^\wedge(n\pi)}{\phi^\wedge(n\pi/d_j)} = \frac{\phi^\wedge(n\pi)}{\phi^\wedge(n\pi/d_j)}.$$

For the Mth order B-wavelets this ratio is

$$\begin{aligned}
\frac{\phi_M^\wedge(n\pi)}{\phi_M^\wedge(n\pi/d_j)} = \frac{\phi_M^\wedge(d_jk\pi)}{\phi_M^\wedge(k\pi)} &= \left(\frac{\phi_1^\wedge(d_jk\pi)}{\phi_1^\wedge(k\pi)}\right)^M \\
&= \left(\frac{1 - z(2d_jk\pi)}{id_jk\pi}\right)^M \left(\frac{1 - z(2k\pi)}{ik\pi}\right)^{-M} \\
&= d_j^{-M}.
\end{aligned}$$

The support of the Mth order B-wavelet is $[0, 2M-1)$. Hence the coefficients Γ_j associated with the Mth order B-wavelets are

$$\Gamma_j = \frac{d_j^{1-M}}{(2M-1)(d_j-1)+1} \ .$$

Figures 1, 2, and 3 show the original wavelets and the first two refinements together with the corresponding filter characteristics.

Example 3.3 *Choosing $J = 1$ implies that no modification takes place and $\psi^1 = \psi$, whereas choosing $J = 2$ implies $d_2 = 3$, $d = 3$. Hence, the enhanced Haar wavelet ψ_1^2 is*

$$\psi_1^2 := \psi_1 - \frac{1}{3}\Big(\psi_1(3\cdot) + \psi_1(3\cdot - 1) + \psi_1(3\cdot - 2)\Big).$$

Choosing $J = 3$ gives $d_2 = 3, d_3 = 5$, and $d = 15$. And so on.

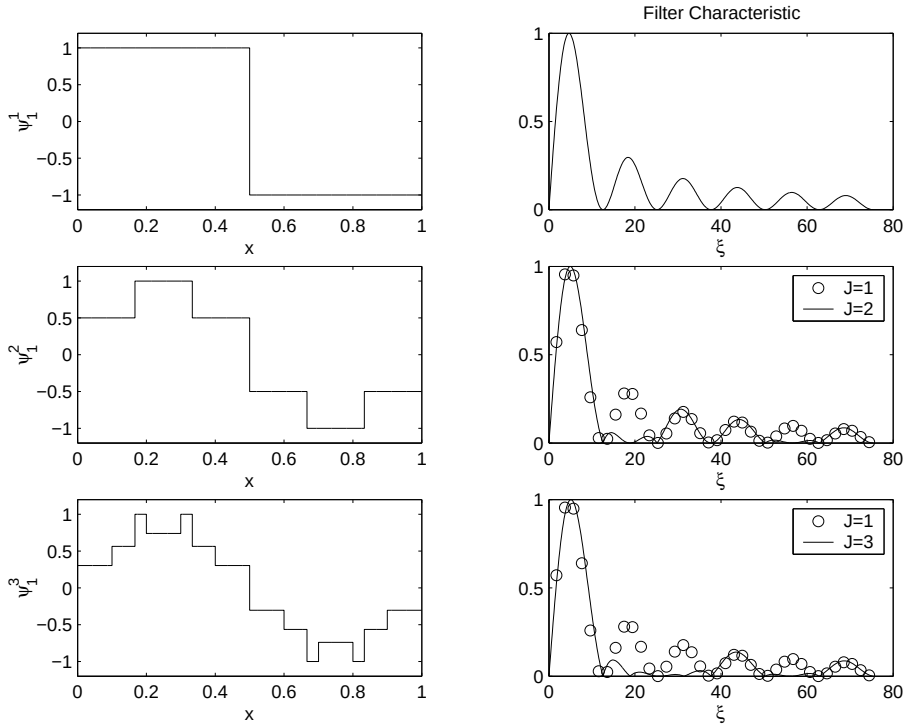

Fig. 1. Left: The modified Haar wavelet ψ_1^J, for $J = 1, 2$, and 3. Right: The filter characteristics $\left|\psi_1^{J\wedge}\right|$, for $J = 1, 2$, and 3.

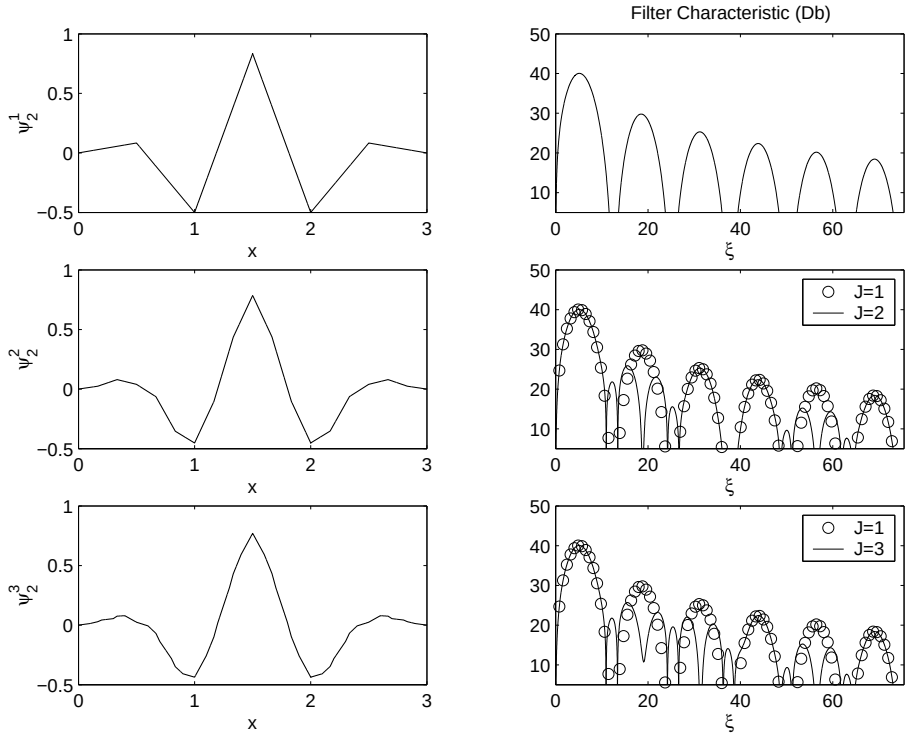

Fig. 2. Left: The modified linear B-spline wavelet ψ_2^J, for $J = 1, 2$, and 3. Right: The filter Characteristics $\left|\psi_2^{J\wedge}\right|$, for $J = 1, 2$, and 3, measured in Db.

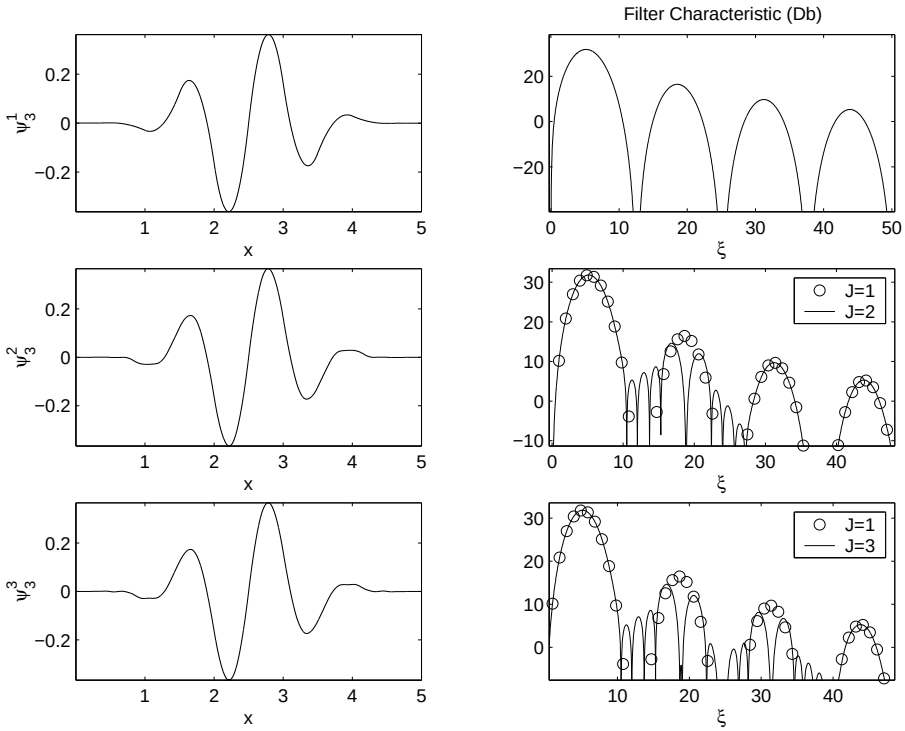

Fig. 3. Left: The modified quadratic B-spline wavelet ψ_3^J, for $J = 1, 2$, and 3. Right: The filter Characteristics $\left|\psi_3^{J\wedge}\right|$, for $J = 1, 2$, and 3, measured in Db.

3.3 Frequency characteristic measures for the refined B-wavelet bases

The enhanced filter characteristic of the modified B-wavelet bases is now illustrated by calculating the measures from Definitions 2.4 and 2.5.

The filter characteristic of the original B-wavelets have zeros at $\xi = 4n\pi$, for $n \in \mathbb{Z}$. By virtue of Theorem 3.2, the modified basis ψ_M^J has extra zeros at $2n\pi$, for $1 < n \leq d_J$.

We compare the filter characteristics $|\psi_M{}^\wedge(\xi)|$ and $\left|\psi_M^{J\wedge}(\xi)\right|$ by considering their power and maxima over the intervals $\Omega_N = (4\pi, 4(N{+}1)\pi)$. The measures are calculated for the original and refined versions of the Haar, linear, and quadratic B-wavelets, and are presented in Tables 1 – 6. For completeness,

N	1	2	3	4	5	6
$\mathcal{M}_N^*\psi_1^1$	5.28	6.26	7.00	7.59	8.09	8.52
$\mathcal{M}_N^*\psi_1^2$	12.32	9.03	8.94	10.00	10.13	10.32
$\mathcal{M}_N^*\psi_1^3$	10.38	12.10	10.63	11.63	11.40	11.36

Table 1
Maxima side lobe/main lobe ratios (in -Db) for ψ_1^J, $J = 1, 2, 3$.

N	1	2	3	4	5	6
$\mathcal{M}_N^* \psi_2^1$	10.26	11.94	13.16	14.11	14.89	15.55
$\mathcal{M}_N^* \psi_2^2$	14.18	14.46	15.30	16.39	17.05	17.63
$\mathcal{M}_N^* \psi_2^3$	14.37	15.81	16.34	17.39	17.97	18.50

Table 2
Maxima side lobe/main lobe ratios (in -Db) for ψ_2^J, $J = 1, 2, 3$.

N	1	2	3	4	5	6
$\mathcal{M}_N^* \psi_3^1$	15.38	17.56	19.05	20.18	21.08	21.84
$\mathcal{M}_N^* \psi_3^2$	17.22	19.01	20.40	21.56	22.44	23.18
$\mathcal{M}_N^* \psi_3^3$	17.19	19.33	20.69	21.85	22.72	23.46

Table 3
Maxima side lobe/main lobe ratios (in -Db) for ψ_3^J, $J = 1, 2, 3$.

N	1	2	3	4	5	6	∞
$\mathcal{M}_{\Omega_N} \psi_1^1$	10.91	9.59	9.05	8.76	8.57	8.44	7.74
$\mathcal{M}_{\Omega_N} \psi_1^2$	26.42	14.42	12.80	12.78	12.25	11.91	10.85
$\mathcal{M}_{\Omega_N} \psi_1^3$	23.26	22.76	16.32	16.29	15.16	14.45	12.39

Table 4
Side-lobe/main lobe power ratios (in -Db) for ψ_1^J, $J = 1, 2, 3$.

N	1	2	3	4	5	6	∞
$\mathcal{M}_{\Omega_N} \psi_2^1$	20.87	20.35	20.23	20.18	20.17	20.15	20.14
$\mathcal{M}_{\Omega_N} \psi_2^2$	29.01	26.39	25.92	25.90	25.82	25.78	25.75
$\mathcal{M}_{\Omega_N} \psi_2^3$	29.07	28.30	27.58	27.55	27.44	27.38	27.31

Table 5
Side-lobe/main lobe power ratios (in -Db) for ψ_2^J, $J = 1, 2, 3$.

N	1	2	3	4	5	6	∞
$\mathcal{M}_{\Omega_N} \psi_3^1$	31.11	30.92	30.89	30.89	30.89	30.89	30.89
$\mathcal{M}_{\Omega_N} \psi_3^2$	35.86	35.31	35.25	35.24	35.24	35.24	35.24
$\mathcal{M}_{\Omega_N} \psi_3^3$	35.84	35.62	35.55	35.54	35.54	35.53	35.53

Table 6
Side-lobe/main lobe power ratios (in -Db) for ψ_3^J, $J = 1, 2, 3$.

the measures $\mathcal{M}$ over $\mathbb{R}_*^+$ are also given.

The tables offer clear evidence that the refinement technique significantly improves the filter characteristic of the B-wavelets about an arbitrarily large localised band-region. In fact, an improvement can be seen over the entire frequency domain.

For the Haar and quadratic wavelet, it can be seen that ψ^3 has slightly poorer performance than ψ^2 over the interval Ω_1. Generally, however, the filter characteristics of ψ^J will improve with larger J.

4 Fast integral wavelet transforms

The integral transform, given by Equation (1) can be written as the convolutions $(\mathcal{W}_\psi f)(\alpha, \tau) = (f * \Psi_\alpha)(\tau)$. In practice the transform is calculated over a discrete set of α, and τ. For N data points, a verbatim calculation of (1) incurs $O(N^2)$ computational complexity per scale. Alternatively, via the fast Fourier transform, $(\mathcal{W}_\psi f)(\alpha, \tau) = (f^\wedge \Psi_\alpha^\wedge)^\vee(\tau)$ requires $O(N \log N)$ complexity per scale.

Of course, wavelets obtained from a multiresolution analysis can be calculated over the dyadic lattice $(\alpha_j, \tau_k) = (2^j, 2^j k)$, for $j, k \in \mathbb{Z}$ with an overall complexity of $O(N)$, via Mallat's fast wavelet transform algorithm [10]. Likewise, zero-padding the input samples and using the algorithme à trous [9] returns the wavelet coefficients over the semi-dyadic lattice $(2^j, k)$ with $O(N)$ complexity per scale.

Restricting the choice of wavelet to those generated by B-splines, Unser et al. [14] proposed an algorithm for the integer lattice $(\alpha_j, \tau_k) = (j, k)$ with $O(N)$ complexity per scale. This result was extended recently by Muñoz et al. [11] for arbitrary scales, again with $O(N)$ complexity per scale.

A brief discussion of these fast integral wavelet transforms, as derived in [14] for integer scales, and [11] for arbitrary scales, will be given below. This is followed by the application of the fast transform to our modified wavelet, defined by (2). Both Unser et al. and Muñoz et al. used splines centred about the origin. However, their theory is easily adapted to the so called non-causal splines that are defined over the positive half of the real line.

The fast algorithm of Unser et al. [14] is facilitated by the following self similarity property of the B-splines.

Proposition 4.1 *(Unser et al.) For $\alpha \in \mathbb{Z}$, there exists a sequence $\{\eta_M^\alpha(k)\}_k$ in $\ell_2(\mathbb{Z})$, such that*

$$\phi_M(\alpha^{-1}\cdot) = \sum_{k\in\mathbb{Z}} \eta_M^\alpha(k)\phi_M(\cdot - k).$$

Proof Clearly $\phi_1(\alpha^{-1}\cdot) = \sum_{k=0}^{\alpha-1} \phi_1(\cdot - k) =: \eta_1^\alpha * \phi_1$ with $\eta_1^\alpha(k) = 1$, for all $0 \le k < \alpha$. Define $\eta_M^\alpha := \alpha^{-1}\eta_1^\alpha * \eta_{M-1}^\alpha$. An $(M-1)$-fold convolution of both sides completes the proof. ■

It follows from Proposition 4.1 and the two-scale relation (5) that

$$\psi_M(\alpha^{-1}\cdot) = \sum_{k,\ell\in\mathbb{Z}} g_M(\ell)\eta_M^\alpha(k)\phi_M(2\cdot - k - \alpha\ell),$$

that is

$$\psi_M(\alpha^{-1}\cdot) = \sum_{k\in\mathbb{Z}} \left(g_M^{\uparrow\alpha} * \eta_M^\alpha\right)(k)\phi_M(2\cdot - k),$$

where the symbol $\uparrow$ indicates the up-sampling operator:

$$g^{\uparrow m}(n) = \begin{cases} g(n/m), & m \mid n \\ 0, & \text{otherwise} \end{cases}, \quad m, n \in \mathbb{Z}.$$

Hence, the integral wavelet transform of f, evaluated over the integer lattice (α, τ) with $\alpha, \tau \in \mathbb{Z}$, can be written as

$$(\mathcal{W}_{\psi_M} f)(\alpha, \tau) = (f * \Psi_{\alpha;M})(\tau) = \left(g_M^{\uparrow\alpha} * \eta_M^\alpha * \langle f, \phi_M(2\cdot - k)\rangle_{L_2}\right)(\tau).$$

Unser et al. [15] propose the following algorithm:

- Perform the initialisation $f_M^1(k) = \langle f, \phi_M(2\cdot - k)\rangle_{L_2}$.
- For each $\alpha \in \mathbb{N}$, calculate $f_M^\alpha = \eta_M^\alpha * f_M^1 = \alpha^{-M} f_M^1 \underset{j=1}{\overset{M}{*}} \eta_1^\alpha$ where each moving average operation $(\eta_1^\alpha * \cdot)$ is implemented recursively via:

$$f_M^\alpha(k) = f_M^\alpha(k-1) + \frac{f_M^{\alpha-1}(k+\alpha) - f_M^{\alpha-1}(k)}{\alpha}.$$

- $(\mathcal{W}_{\psi_M} f)(\alpha, \tau) = \left(g_M^{\uparrow\alpha} * f_M^\alpha\right)(\tau)$.

In such a manner, each point $f_\alpha(k)$ can be calculated with $2M$ additions. Since the sequence g_M is symmetric or anti-symmetric and has length $3M-1$, the final convolution will require $\lceil (3M-1)/2 \rceil$ multiplications and $3M-2$ additions per value of τ.

4.2 Arbitrary scale B-wavelets

The algorithm of Muñoz et al. [11] admits arbitrary scales. They define Δ_M as the M-th fold finite difference operator:

$$\Delta_M := \sum_{k=0}^{M} \binom{M}{k} (-1)^k \delta(\cdot - k),$$

where δ is Dirac's delta. Accordingly, Δ_{-M} denotes the inverse of Δ_M and is equivalent to the Mth iteration of $\Delta_{-1} = \sum_{n \geq 0} \delta(x-n)$. The values

$$w_M^\alpha = \sum_k \left(g_M(\ell) * \binom{M}{\ell} (-1)^\ell \right) (k) \delta(\cdot - \alpha k) * \phi_M(2\cdot)$$

are calculated prior to run time and stored in a look table. Denoting $\widetilde{\phi}_M$ as the scaling function, biorthonormal to ϕ_M, the initialisation stage

$$c_M(k) = \Delta_{-M} * \langle f, \widetilde{\phi}_M(2 \cdot - k) \rangle_{L_2},$$

is followed by the calculation

$$(\mathcal{W}_{\psi_M} f)(\alpha, \tau) = w_M^\alpha * c_M,$$

for each α in some subset of $\mathbb{R}$. Although this algorithm is not quite as efficient as the first for integer scales, it still manages $O(N)$ complexity per scale and is obviously more versatile.

4.3 Fast algorithms for the modified B-wavelet

The fast integral wavelet transform algorithms can be adapted for the modified B-wavelets. Since the modified wavelets comprise sums of integer dilated and shifted versions of the original wavelet, the adaptation is quite natural. Recall that the modified B-wavelet is given as

$$\psi_M^J = \sum_{j=1}^{J} \mu(d_j) \Gamma_j \sum_{n=0}^{X(d_j-1)} \psi_M(d_j \cdot - n).$$

With respect to the dilated B-splines, this is

$$\psi_M^J = \sum_{j=1}^{J} \mu(d_j)\Gamma_j \sum_{k\in\mathbb{Z}} g_{M;j}(k)\phi_M(2d_j\cdot -k), \tag{8}$$

with

$$g_{M;j}(k) := \sum_{n=0}^{X(d_j-1)} g_M(k-2n).$$

For integer scales $\alpha \in \mathbb{Z}$, we note that

$$\phi_M(d_j\alpha^{-1}\cdot) = \sum_{k\in\mathbb{Z}} \eta_M^\alpha(k)\phi_M(d_j\cdot -k)$$

is a corollary of Proposition 4.1. From the definition of the modified wavelet (2), it follows that

$$\psi_M^J(\alpha^{-1}\cdot) = \sum_{j=1}^{J} \mu(d_j)\Gamma_j \sum_{k\in\mathbb{Z}} \left(g_{M;j}^{\uparrow\alpha} * \eta_M^\alpha\right)(k)\phi_M(2d_j\cdot -k),$$

and similar to the original scheme, the algorithm is initialised with

$$f_{M;j}^1(k) = \langle f, \phi_M(2d_j\cdot -k)\rangle_{L_2},$$

for $j = 1,\ldots,J$. Next, $f_{M;j}^\alpha = \eta_M^\alpha * f_{M;j}^1$ is calculated recursively. Finally, for $\alpha,\tau\in\mathbb{Z}$

$$(\mathcal{W}_{\psi_M} f)(\alpha,\tau) = \sum_{j=1}^{J} \mu(d_j)\Gamma_j \left(g_{M;j}^{\uparrow\alpha} * f_{M;j}^\alpha\right)(\tau).$$

Although clearly slower than the unmodified B-wavelet case, the algorithm still retains $O(N)$ complexity per scale. Similarly, for arbitrary scales $\alpha\in\mathbb{R}$ we put

$$w_{M;j}^\alpha := \sum_{j=1}^{J} \mu(d_j)\Gamma_j \sum_k \left(g_{M;j}(\ell) * \binom{M}{\ell}(-1)^\ell\right)(k)\delta(\cdot-\alpha k) * \phi_M(2d_j\cdot)$$

and calculate the initialisation stage

$$c_M(k) = \Delta_{-M} * \langle f, \widetilde{\phi}_M(2\cdot -k)\rangle_{L_2},$$

followed by

$$(\mathcal{W}_{\psi_M} f)(\alpha,\tau) = w_M^\alpha * c_M.$$

Again, this is slower than the original algorithm but shares the $O(N)$ complexity per scale property.

5 Multi-scaling functions and the modified wavelets

The construction of a multiresolution analysis (MRA) not only facilitates wavelet design but also raises the opportunity to perform fast discrete wavelet transforms via pyramidal algorithms [10]. Early multiresolution analyses were generated by dyadic dilations and translations of a single scaling function $\phi \in L_2(\mathbb{R})$. A natural extension of this theory to the case of multi-scaling functions $\boldsymbol{\phi} = (\phi^0, \ldots, \phi^{R-1})^T$, with $\phi^r \in L_2(\mathbb{R})$, for $r = 0, \ldots, R-1$, was pioneered by Alpert [1], Geronimo et al. [6], and Goodman et al. [7], [8]. Accordingly, R-many scaling functions and R-many wavelet functions are used to construct a basis for L_2.

Multi-wavelets hold certain advantages over single, so called scalar, wavelets. Daubechies [4] proved that an orthogonal, finitely supported, symmetric, scalar wavelet could not have an approximation order greater than one. However, Geronimo et al. [6] managed to circumvent this restriction by considering a multiwavelet design.

Like the scalar case, the multi-scaling functions give rise to a fast, pyramidal wavelet transform algorithm. This motivates us to place the modified wavelets into the setting of a MRA, generated by a multi-scaling functions. To this end, the following notation will be useful:

$$\begin{aligned}
\ell_2^R(\mathbb{Z}) &= \left\{ \mathbf{c} = (c_0, \ldots, c_{R-1})^T : \; c_j \in \ell_2(\mathbb{Z}),\; 0 \le j < R \right\} \\
\ell_2^{R\times R}(\mathbb{Z}) &= \left\{ \mathbf{C} = (C_{j,k})_{j,k=0}^{R-1} : \; C_{j,k} \in \ell_2(\mathbb{Z}),\; 0 \le j,k < R \right\} \\
L_2^R(\mathbb{R}) &= \left\{ \mathbf{f} = (f_0, \ldots, f_{R-1})^T : \; f_j \in L_2(\mathbb{R}),\; 0 \le j < R \right\}.
\end{aligned}$$

That is, $\ell_2^R(\mathbb{Z})$ comprises R dimensional vectors of square summable sequences, whereas $\ell_2^{R\times R}(\mathbb{Z})$ comprises R-by-R matrices of square summable sequences, and $L_2^R(\mathbb{R})$ comprises R dimensional vectors of square integrable functions.

Definition 5.1 *Let $\boldsymbol{\phi} := (\phi^0, \ldots, \phi^{R-1})^T \in L_2^R(\mathbb{R})$. For $j \in \mathbb{Z}$, define*

$$V_j(\boldsymbol{\phi}) := \operatorname{clos}\operatorname{span}\{2^{j/2}\phi^r(2^j \cdot -k) : 0 \le r < R,\; k \in \mathbb{Z}\}.$$

Then $\boldsymbol{\phi}$ is a multiscaling function that generates the MRA $\{V_j(\boldsymbol{\phi})\}$ of multiplicity R if all of the following properties are satisfied.

(1) $V_j(\boldsymbol{\phi}) \subset V_{j+1}(\boldsymbol{\phi}), \quad \forall j \in \mathbb{Z}$.
(2) $f \in V_j(\boldsymbol{\phi}) \Leftrightarrow f(2\cdot) \in V_{j+1}(\boldsymbol{\phi}), \quad \forall j \in \mathbb{Z}$.
(3) $\operatorname{clos}\bigcup_{j\in\mathbb{Z}} V_j(\boldsymbol{\phi}) = L_2(\mathbb{R})$.
(4) $\bigcap_{j\in\mathbb{Z}} V_j(\boldsymbol{\phi}) = \{0\}$.
(5) The set $\{\phi^r(\cdot - k) : 0 \le r < R,\; k \in \mathbb{Z}\}$ forms a Riesz basis for $V_0(\boldsymbol{\phi})$.

Commonly, the multi-scaling function is used to define a multiwavelet. Here, however, we take a different approach. We construct a scalar wavelet from the multi-scaling function. This allows the modified wavelet to be described in the context of a MRA, with multiplicity R.

Definition 5.2 *The function $\psi \in L_2(\mathbb{R})$ is said to be generated by a MRA $\{V_j(\boldsymbol{\phi})\}$ of multiplicity R, with $\boldsymbol{\phi} = (\phi^0, \ldots, \phi^{R-1})^T \in L_2^R(\mathbb{R})$ if there exists some $\{(g^0(k), \ldots, g^{R-1}(k))^T\}_{k\in\mathbb{Z}} \in \ell_2^R(\mathbb{Z})$, such that*

$$\psi = \sum_{\substack{0\le r<R \\ k\in\mathbb{Z}}} g^r(k)\phi^r(2\cdot -k). \tag{9}$$

Using vector notation, define $\mathbf{g}(k) := (g^0(k), \ldots, g^{R-1}(k))^T$. Then (9) can be written as a natural extension of the two scale relation for scalar functions (5):

$$\psi = \sum_{k\in\mathbb{Z}} \mathbf{g}(k)^T \boldsymbol{\phi}(2\cdot -k).$$

5.1 Modified B-wavelet construction

Recall the Mth order B-spline function ϕ_M and the Mth order modified wavelet ψ_M^J. Define the multi-scaling function $\boldsymbol{\phi}_M^J := (\phi_M^{J,0}, \ldots, \phi_M^{J,d-1})^T$, with $\phi_M^{J,r} := \phi_M(d\cdot -r)$, where $d_{j+1} = \min_{n\in 2\mathbb{N}-1}\left\{n > d_j \text{ and } \max_{n^\alpha | d}\alpha = 1\right\}$ for $j = 2, \ldots, J$, and $d_1 := 1$, and $d := \prod_{j=1}^J d_j$.

Furthermore, recall that $\widetilde{\phi}_M$ is the scaling function, biorthonormal to ϕ_M. That is $\langle \phi_M(\cdot - m), \widetilde{\phi}_M(\cdot - n)\rangle = \delta_{m,n}$. Some results are now established for the modified B-wavelets in the setting of a MRA of multiplicity d. This will expedite the pyramidal algorithm, discussed below in Section 5.3.

Theorem 5.3 *There exists a d-by-d matrix of sequences $\mathbf{H}_M^J \in \ell_2^{d\times d}(\mathbb{Z})$ such that $\boldsymbol{\phi}_M^J = \sum_{k\in\mathbb{Z}} \mathbf{H}_M^J(k)\boldsymbol{\phi}_M^J(2\cdot -k)$.*

Proof For the B-splines there exists $\{h_M(k)\}_{k\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$ such that

$$\phi_M = \sum_{k\in\mathbb{Z}} h_M(k)\phi_M(2\cdot -k).$$

That is

$$\begin{aligned}\phi_M^{J,r} &= \sum_{k\in\mathbb{Z}} h_M(k)\phi_M(2d\cdot -2r-k)\\ &= \sum_{k\in\mathbb{Z}}\sum_{s=0}^{d-1} h_M(s-2r+kd)\phi_M^{J,s}(2\cdot -k).\end{aligned}$$

Define the sth column and rth row of $\mathbf{H}_M^J(k)$ as $h_M(s-2r+kd)$, such that $h_M(k)=0$ for $k<0$ and $k>M$. ■

Example 5.4 *Choose $J=2$, and $M=1$. For then, $\boldsymbol{\phi}_1^2=(\phi_1^{2,1})_{j=0}^2$ with $\phi_1^{2,j}:=\phi_1(3\cdot -j)$ and*

$$\mathbf{H}_1^2(0)=\begin{pmatrix}1&1&0\\0&0&1\\0&0&0\end{pmatrix},\quad \mathbf{H}_1^2(1)=\begin{pmatrix}0&0&0\\1&0&0\\0&1&1\end{pmatrix}.$$

Theorem 5.5 *There exists a vector of functions $\widetilde{\boldsymbol{\phi}}_M^J:=((\widetilde{\phi}_M^{J,r}(\cdot-n))_{r=0}^{d-1})^T$ in $L_2^d(\mathbb{R})$, with $V_0(\widetilde{\boldsymbol{\phi}}_M^J)\supset V_0(\widetilde{\phi}_M)$ such that, for $f\in V_0(\widetilde{\phi}_M)$ we have*

$$f=\sum_{k\in\mathbb{Z}}(\widetilde{\boldsymbol{\phi}}_M^J(\cdot-k))^T(\mathcal{P}_M^J f)(k),$$

where the mapping $\mathcal{P}_M^J: V_0\Big(\widetilde{\phi}_M\Big)\mapsto \ell_2^d(\mathbb{Z})$ is defined by

$$\Big\{\Big(\mathcal{P}_M^J f\Big)(k)\Big\}_{k\in\mathbb{Z}}:=\Big\{\Big(\Big\langle f,\phi_M^{J,0}(\cdot-k)\Big\rangle,\ldots,\Big\langle f,\phi_M^{J,d-1}(\cdot-k)\Big\rangle\Big)^T\Big\}_{k\in\mathbb{Z}}.$$

Proof Firstly, we prove that $\mathcal{P}_M^J$ is a bounded and boundedly invertible linear operator. From Proposition 4.1 it follows that there exists $\eta_M^d\in\ell_2(\mathbb{Z})$ such that

$$\begin{aligned}\phi_M &= \sum_{k\in\mathbb{Z}}\eta_M^d(k)\phi_M(d\cdot -k)\\ &= \sum_{k\in\mathbb{Z}}\sum_{r=0}^{d-1}\eta_M^d(r+kd)\phi_M^{J,r}(\cdot-k).\end{aligned}$$

Therefore, $f=\sum_{n\in\mathbb{Z}}\langle f,\phi_M(\cdot-n)\rangle\widetilde{\phi}_M(\cdot-n)\in V_0(\widetilde{\phi}_M)$ can be expressed as

$$f=\sum_{n,k\in\mathbb{Z}}\widetilde{\phi}_M(\cdot-n)\Big(\eta_M^d(d(k-n)),\ldots,\eta_M^d(d(k-n+1)-1)\Big)(\mathcal{P}_M^J f)(k).$$

Now, clearly $\widetilde{\phi}_M^{J,r}=\sum_{n\in\mathbb{Z}}\eta_M^d(r-nd)\widetilde{\phi}_M(\cdot-n)\in V_0(\widetilde{\phi}_M)$. ■

Theorem 5.6 *There exists a sequence* $\mathbf{g}_M^J \in \ell_2^d(\mathbb{Z})$ *such that the two scale relation* $\psi_M^J = \sum_{k\in\mathbb{Z}} \mathbf{g}_M^J(k)^T \boldsymbol{\phi}_M^J(2\cdot - k)$ *holds.*

Proof From (8) we have:

$$\psi_M^J = \sum_{j=1}^{J}\sum_{k\in\mathbb{Z}} \mu(d_j)\Gamma_j g_{M;j}(k)\phi_M(2d_j\cdot - k).$$

Proposition 4.1 implies $\phi_M(d_j\cdot) = \sum_{k=0}^{M(d_j^*-1)} \eta_M^{d_j^*}(n)\phi_M(d\cdot - k)$, with $d_j^* := d/d_j$. Hence

$$\psi_M^J = \sum_{j=1}^{J}\sum_{k\in\mathbb{Z}}\sum_{n=0}^{M(d_j^*-1)} \mu(d_j)\Gamma_j g_{M;j}(k)\eta_M^{d_j^*}(n)\phi_M(2d\cdot - n - kd_j).$$

Put $r + qd := n + kd_j^*$, with $0 \le r < d$. That is $q = q_j(n,k) := \lfloor d^{-1}(n + kd_j^*)\rfloor$, and $r = r_j(n,k) := n + kd_j - qd$. We now have

$$\psi_M^J = \sum_{j=1}^{J}\sum_{k\in\mathbb{Z}}\sum_{n=0}^{M(d_j^*-1)} \mu(d_j)\Gamma_j g_{M;j}(k)\eta_M^{d_j^*}(n)\phi_M^{J,r_j(n,k)}(2\cdot - q_j(n,k)),$$

which is of the form $\psi_M^J = \sum_{q,r} g_M^{J,r}(q)\phi_M^{J,r}(2\cdot - q)$. ■

Example 5.7 *Choosing* $J = 2$ *and* $M = 1$, *yields* $\mathbf{g}_1^2(0) = (1,2,1)^T$, *and* $\mathbf{g}_1^2(1) = -(1,2,1)^T$.

5.2 Multi-wavelet algorithms

Mallat's scalar wavelet pyramidal algorithm was extended to the multiwavelet case by Xia et al [16]. A brief synopsis of the algorithm is given before we adapt the idea to the modified B-wavelet.

Consider the multi-scaling functions $\{\boldsymbol{\phi}_{j,n}\} := \{(2^{j/2}\phi^r(2^j\cdot - n))_{r=0}^{R-1}\}$, the multiwavelet functions $\{\boldsymbol{\psi}_{j,n}\} := \{(2^{j/2}\psi^r(2^j\cdot - n))_{r=0}^{R-1}\}$, and their two scale equations given by

$$\begin{aligned}\boldsymbol{\phi}_{j-1,n} &= \sum_{k\in\mathbb{Z}} \mathbf{H}(k-2n)\boldsymbol{\phi}_{j,k}\\ \boldsymbol{\psi}_{j-1,n} &= \sum_{k\in\mathbb{Z}} \mathbf{G}(k-2n)\boldsymbol{\phi}_{j,k},\end{aligned}$$

where $\mathbf{H}, \mathbf{G} \in \ell_2^{R\times R}(\mathbb{Z})$. Let $\{\boldsymbol{\lambda}_{j,n}\}_{n\in\mathbb{Z}}, \{\boldsymbol{\gamma}_{j,n}\}_{n\in\mathbb{Z}} \in \ell_2^R(\mathbb{Z})$. A function f in

$V_0(\widetilde{\boldsymbol{\phi}})$, can be expressed as

$$f = \sum_{k\in\mathbb{Z}} \boldsymbol{\lambda}_{0,k}^T \widetilde{\boldsymbol{\phi}}_{0,k},$$

where the rth element of $\boldsymbol{\lambda}_{0,k}$ is found by calculating $\langle f, \phi^r(2\cdot - k)\rangle_{L_2}$. Accordingly, the coefficients $\{\boldsymbol{\lambda}_{j,n}\}_{j<0}$ of f at the larger scales can be calculated by recursively applying

$$\boldsymbol{\lambda}_{j-1,n} = \sum_{k\in\mathbb{Z}} \overline{\mathbf{H}(k-2n)}\boldsymbol{\lambda}_{j,k}.$$

Similarly, the associated wavelet decomposition coefficients $\{\boldsymbol{\gamma}_{j,n}\}_{j<0}$ of f can be calculated recursively via

$$\boldsymbol{\gamma}_{j-1,n} = \sum_{k\in\mathbb{Z}} \overline{\mathbf{G}(k-2n)}\boldsymbol{\lambda}_{j,k}.$$

5.3 Multi-B-wavelet recursive algorithm

Let $\mathbf{H} \in \ell_2^{R\times R}(\mathbb{Z})$, $\{\boldsymbol{\lambda}_{j,n}\}_{n\in\mathbb{Z}}, \mathbf{g} \in \ell_2^R(\mathbb{Z})$, and $\{\gamma_{j,n}\}_{n\in\mathbb{Z}} \in \ell_2(\mathbb{Z})$. We define $\boldsymbol{\phi}_{M;j,n}^J := ((2^{j/2}\phi_M^{J,r}(2^j\cdot - n))_{r=0}^{R-1})^T$, and $\widetilde{\boldsymbol{\phi}}_{M;j,n}^J := ((2^{j/2}\widetilde{\phi}_M^{J,r}(2^j\cdot - n))_{r=0}^{R-1})^T$. For $f \in V_0(\widetilde{\phi}_M)$, Theorems 5.3 and 5.5 imply that we can write

$$f = \sum_{k\in\mathbb{Z}} \boldsymbol{\lambda}_{0,k}^T \widetilde{\boldsymbol{\phi}}_{M;0,k}^J,$$

with

$$\boldsymbol{\lambda}_{0,k} = \Big(\langle f, \phi_M^{J,0}(2\cdot - k)\rangle, \ldots, \langle f, \phi_M^{J,d-1}(2\cdot - k)\rangle\Big)^T, \tag{10}$$

such that the two scale relation

$$\boldsymbol{\lambda}_{j-1,n} = \sum_{k\in\mathbb{Z}} \overline{\mathbf{H}_M^J(k-2n)}^T \boldsymbol{\lambda}_{j,k} \tag{11}$$

is satisfied. From Theorem 5.6 it follows that

$$\gamma_{j-1,n} = \sum_{k\in\mathbb{Z}} \overline{\mathbf{g}_M^J(k-2n)}^T \boldsymbol{\lambda}_{j,k}. \tag{12}$$

Herewith the possibility arises to implement a fast discrete multiwavelet transform for our modified wavelets. The coefficients are initialised by (10). The $\boldsymbol{\lambda}_{j,n}$ are computed recursively with (11). Consequently, the wavelet coefficients are found via (12).

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