A multi-channel multi-sampling rate theorem.

James D.B. Nelson
Applied Mathematics and Computing
Cranfield University
j.nelson@cranfield.ac.uk

Abstract

The Möbius function is utilised to propose a class of multi-channel, multi-sampling rate theorems. Our construction facilitates a parallel decomposition of the Fourier transform into an array of general integral transform operators. A practical example is discussed which employs a complex square wave biorthonormal basis.

Mathematics Subject Classification 94A20

1 Introduction

A signal with power restricted to some bounded base-band region Ω, given by $\Omega = (-\pi/\tau, \pi/\tau)$ lends itself to the following well known cardinal series

$$f(t) = \sum_n f(n\tau)\,\mathrm{sinc}\,\frac{1}{\tau}(t - n\tau). \tag{1}$$

The crux of this result is that the signal $f(t)$ can be represented and recovered by its samples $\{f(n\tau)\}$. Names commonly associated with the cardinal series include Kotel'nikov [11], Shannon [15], Someya [16], and Whittaker [18].

A natural extension to (1) is a series for a function with non-zero power content restricted to a pass-band, rather than base-band, region. Kohlenberg [10] realized this series by using the function samples $\{f(n\tau)\}$, together with a set of translated samples of the function $\{f(n\tau + \sigma)\}$. His idea was later formalised in the Riesz basis setting by Higgins [6]. Alternatively, Woodward [19] and Goldman [5] used the samples of the function together the samples of its Hilbert transform and Butzer, Splettstößer, and Stens [3] give an example which uses the samples of a function and its derivative.

A further extension is to consider a function with non-zero power content restricted to a collection of disjoint band-regions. This 'multi-band' problem was tackled by Bezuglaya and Katsnelson [2] for the case where all of the sub-bands and gaps are commensurable. Higgins [7], [8] developed this theory in the multi-channel context. The resulting multi-channel series comprises a set of translated uniform samples $\{f(n\tau + \sigma_j)\}_{j,n}$.

Papoulis [14] explored multi-channel methods for base-band signals and discovered a common theme where each of the N channels contained a function $g_n(t)$, associated with the signal via a time invariant linear filter. The

g_n are sampled at $1/N$ times the Nyquist sampling rate, thus yielding an overall sampling density comparable to that of (1). These extensions to the original cardinal series became known as general sampling expansions.

In particular, the samples of a function and its first N derivatives give rise to a generalised sampling expansion. The expansion for the case $N = 1$ was first given by Jagerman and Fogel [9] and the general case by Linden and Abramson [12]. Some more recent efforts have focused upon the problem of non-uniform sampling theory [1], [13].

The generalised sampling theorems utilise operational properties of the Fourier transform. Three of the operational properties exploited in some of the examples discussed above are the shift/modulation property

$$(f(t+\sigma))^\wedge(\omega) = e^{i\sigma\omega} f^\wedge(\omega),$$

the derivative property

$$\left(\frac{d^n}{dt^n} f(t)\right)^\wedge(\omega) = i^n \omega^n f^\wedge(\omega),$$

and the Hilbert transform property

$$(Hf)^\wedge(\omega) = -i \operatorname{sgn} \omega f^\wedge(\omega),$$

where $f^\wedge$ denotes the Fourier transform of f, and Hf denotes the Hilbert transform of f. A multi-channel, multi-sampling rate theorem is introduced in Theorem 4.1, below. Unlike the series discussed above, it will be shown that this theorem is expedited by the Möbius arithmetic function, taken from the field of number theory.

2 Preliminaries

For the sake of generality and convenience, use will be made of Hilbert spaces. These are linear inner product spaces, complete under the norm governed by the inner product.

Definition 2.1 *If $\mathcal{H}$ is a Hilbert space with inner product $\langle ., . \rangle$, then a subset $\{e_n\} \in \mathcal{H}$ is orthonormal if $\langle e_m, e_n \rangle = \delta_{m,n}$, where $\delta_{m,n}$ denotes the Kronecker delta. Furthermore, if it is also true that $\{e_n\}$ spans $\mathcal{H}$, then $\{e_n\}$ is an orthonormal basis.*

Separable Hilbert spaces are of interest here because they contain a complete orthonormal basis $\{e_n\}$ such that any $f \in \mathcal{H}$ can be expressed as

$$f = \sum_n \langle f, e_n \rangle e_n. \tag{2}$$

From the series expansion (2) it can be observed that the elements which arise from the inner product $\langle f, e_n \rangle$ are precisely the coefficient functionals that express the function f in terms of the basis $\{e_n\}$. To explore more general forms of this representation, it is expedient to introduce the notion of a Riesz basis.

Definition 2.2 *Let $\{e_n\}$ be an orthonormal basis of $\mathcal{H}$. A subset $\{h_n\} \in \mathcal{H}$ is a Riesz basis if there is a bounded and boundedly invertible linear operator T of $\mathcal{H}$ such that $Te_n = h_n$ for all n.*

The notion of an orthonormal basis can be generalised to allow utility to be shared between two associated bases.

Definition 2.3 *Let $\{h_n\}$ and $\{h_n^*\}$ be two subsets in the Hilbert space $\mathcal{H}$. Then the set $\{h_n, h_n^*\}$ constitutes a biorthonormal system for $\mathcal{H}$ if $\langle h_n, h_m^* \rangle = \delta_{n,m}$.*

If $\{h_n\}$ is a basis for $\mathcal{H}$, then so is $\{h_n^*\}$. Such bases $\{h_n\}$ and $\{h_n^*\}$ are Riesz bases and are said to be duals of each other. We denote the adjoint of T as the operator T^* such that $\langle Tf, g \rangle = \langle f, T^*g \rangle$ for all $f, g \in \mathcal{H}$. Now, since

$$\langle h_n, (T^{-1})^* e_m \rangle = \langle T^{-1} h_n, e_m \rangle = \langle e_n, e_m \rangle = \delta_{n,m} \quad ,$$

it follows that $(T^{-1})^* e_n = (TT^*)^{-1} h_n$ is the dual basis of h_n. Furthermore, we can express $(TT^*)f$ as

$$(TT^*)f = T(T^*)f = T \sum_n \langle T^* f, e_n \rangle e_n = \sum_n \langle f, Te_n \rangle Te_n.$$

Hence, f can be expanded in terms of either of the dual bases:

$$f = \sum_n \langle f, h_n \rangle h_n^* = \sum_n \langle f, h_n^* \rangle h_n.$$

Any pair of biorthonormal bases have this property. Without loss of generality, the basis $\{h_n\}$ is used to analyse f by taking the inner product $f^{\sim} := \langle f, h_n \rangle$. Conversely, the basis $\{h_n^*\}$ can be used to reconstruct, or synthesise, $f^{\sim}$ by projecting it back to the domain of f via the calculation $f = \langle f^{\sim}, h_n^* \rangle$. In general, once the h_n have been constructed, it is difficult to derive the dual basis $\{h_n^*\}$ explicitly. This is owing to the difficulty of finding a closed form expression for the inverse of T. However, a derivation is given below which can be used for a class of biorthonormal systems.

3 Construction

Let f belong to the class of all Lebesgue square integrable functions, i.e. $f \in L^2(\mathbb{R})$. Then the following definitions apply.

Definition 3.1 *Let $f \in L^2(\mathbb{R})$. The Fourier transform of f is defined by*

$$f^\wedge(\omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-i\omega t} dt.$$

Definition 3.2 *Let $\{h_\omega\} \in L^2(\mathbb{R})$ be a Riesz basis, and let $f \in L^2(\mathbb{R})$. Then define the generalised transform $f^\sim$ of f as*

$$f^\sim(\omega) := \int_{\mathbb{R}} f(t) \overline{h_\omega(t)} dt.$$

Definition 3.3 *Let $f \in L^2(\mathbb{R})$ be periodic with period $2\pi/\omega$, then f can be expressed as the Fourier series:*

$$f(t) = \sqrt{\frac{\omega}{2\pi}} \sum_{n \in \mathbb{Z}} c_n(f) e^{in\omega t},$$

with the Fourier coefficients $\{c_n\}$ defined by

$$c_n(f) = \sqrt{\frac{\omega}{2\pi}} \int_{-\pi/\omega}^{\pi/\omega} f(t) e^{-in\omega t} dt.$$

We construct the Riesz basis functions $\{h_\omega\}$ such that they share symmetry and periodicity properties with the set of exponentials $\{e^{i\omega t}\}$. Accordingly, let the real part of h_ω be even and the imaginary part be odd. Thence, we require

$$\begin{aligned} \Re(h_\omega(t)) &\equiv \Re(h_\omega(-t)), \quad (3) \\ \Im(h_\omega(t)) &\equiv \Im(-h_\omega(-t)). \quad (4) \end{aligned}$$

The $\{h_\omega\}$ are endowed with a periodicity of $2\pi/\omega$, for every ω. That is to say

$$h_\omega(t) \equiv h_\omega(t + 2\pi/\omega), \quad (5)$$

a consequence of which is that the elements of the basis are ordered with respect to scale:

$$h_{n\omega}(t) \equiv h_\omega(nt).$$

Since h_ω is periodic, it has a Fourier series, namely

$$h_\omega(t) = \sqrt{\frac{\omega}{2\pi}} \sum_{m \in \mathbb{Z}} c_m(h_\omega) e^{im\omega t}.$$

Additional to the symmetry and periodicity constraints, we insist that the ratio

$$\frac{c_{km}(h_n)}{c_k(h_n)} \tag{6}$$

be independent of k. These properties give rise to the following Lemma.

Lemma 3.4 *Let $c_k(\cdot)$ denote the Fourier series coefficient functionals and let $\{h_n\}$ denote a set of periodic functions such that $\Re h_n$ is non-zero and even, and $\Im h_n$ is non-zero and odd. Given that $c_k(h_n) \neq 0$ and that $c_{km}(h_n)/c_k(h_n)$ is independent of k, it follows that $c_{-k}(h_n) = 0$.*

Proof Assume that $c_{-k}(h_n) \neq 0$, then

$$\frac{c_{-km}(h_n)}{c_{-k}(h_n)}$$

exists and is independent of k. This implies that

$$\frac{c_{-km}(h_n)}{c_{-k}(h_n)} = \frac{c_{km}(h_n)}{c_k(h_n)}.$$

But when $m = -1$

$$(c_k(h_n))^2 = (c_{-k}(h_n))^2,$$

i. e.

$$(c_k(h_n))^2 = (c_k(\overline{h_n}))^2,$$

thus contradicting the condition $\Re h_n \neq 0$ or $\Im h_n \neq 0$. $\square$

A family of bases is now created by forming linear combinations of scaled versions of the original basis $\{h_n\}$. We define this process as a mapping $\{h_n\} \mapsto \{h_{n;M}\}$ such that

$$h_{n;M}(t) = \sum_{|m| \leq M} \Gamma_{n,m} h_{mn}(t), \tag{7}$$

where $\Gamma_{n,m}$ is a suitably chosen set of coefficients. The Fourier series expansion of the right hand side of (7) is

$$h_{n;M}(t) = \sqrt{\frac{n}{2\pi}} \sum_{|m| \leq M,\, k \in \mathbb{Z}} \Gamma_{n,m} c_k(h_n) e^{ikmnt}. \tag{8}$$

We assume that the ratio $c_{km}(h_n)/c_k(h_n)$ is independent of k, and, for non-zero $c_k(h_n)$, we write $\Gamma_{n,m}$ as

$$\Gamma_{n,m} = \frac{c_{km}(h_n)}{c_k(h_n)} \mu(|m|), \tag{9}$$

where μ is the Möbius function given by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0, & \text{otherwise} \end{cases}.$$

We put $\ell = km$, and Equation (8) becomes

$$h_{n;M}(t) \quad = \quad \sqrt{\frac{n}{2\pi}} \sum_{\substack{m|\ell \\ |m|\leq M}} \mu(|m|) \sum_{\ell} c_\ell(h_n) e^{i\ell nt}.$$

The Möbius function is employed here due to the utility afforded by the following result taken from number theory.

Theorem 3.5 *Let μ denote the Möbius function. Then*

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{for } n = 1 \\ 0, & \text{otherwise} \end{cases}.$$

Together, Lemma 3.4 and Theorem 3.5 imply that

$$h_{n;M} - \sqrt{\frac{n}{2\pi}} c_1(h_n) e^{int} = \sqrt{\frac{n}{2\pi}} \sum_{\substack{m|\ell \\ 1\leq m\leq M}} \mu(m) \sum_{|\ell|>M} c_\ell(h_n) e^{i\ell nt}, \tag{10}$$

and we observe that, with the exception of $m = 1$, the basis $h_{n;M}$ is devoid of all Fourier coefficients c_m for $m \leq M$. Indeed, by scaling $\{h_n\}$ such that $c_1(h_n) = \sqrt{2\pi/n}$, we find

$$\lim_{M\to\infty} h_{n;M}(t) = e_n = \frac{1}{\sqrt{2\pi}} e^{-int}.$$

Hence, given that $c_{km}(h_n)/c_m(h_n)$ is independent of k, such that h_n satisfies the symmetry conditions (3) and (4), and the periodicity condition (5), the process described by the mapping $\{h_n\} \mapsto \lim_{M\to\infty}\{h_{n;M}\}$ is equivalent to the operation $T^{-1}h_n = e_n$. In this case, Equation (7) reads

$$e_n = \sum_{m\in\mathbb{Z}} \Gamma_{n,m} h_{mn}.$$

Using (9) we have

$$e_n = \sum_{m\in\mathbb{Z}} \mu(|m|) \frac{c_{\ell m}(h_n)}{c_\ell(h_n)} h_{mn}.$$

Again, the assumption that $c_{\ell m}/c_\ell$ is independent of k is asserted. For then we merely put $\ell = 1$. Putting $k = mn$ gives

$$e_n = \sum_{k \in n\mathbb{Z}} \mu(|k/n|) \frac{c_{k/n}(h_n)}{c_1(h_n)} h_k.$$

Now, $c_1(h_n) = n^{-1/2} c_1(h_1)$, and $c_1(h_1)$ can be normalised so that the operator T^{-1} can be written as the matrix

$$T^{-1}_{n,k} = \sqrt{n}\mu(|k/n|)c_{k/n}(h_n). \tag{11}$$

From the Fourier series representation of h_n it is seen that

$$h_n = \sqrt{n} \sum_{k \in n\mathbb{Z}} c_{k/n}(h_n) e_k.$$

Hence T can be associated with the matrix operator $T_{n,k} = \sqrt{n} c_{k/n}(h_n)$, and a comparison with (11) reveals the relation

$$T^{-1}_{n,k} = \mu(|k/n|) \star T_{n,k},$$

where $\cdot \star \cdot$ denotes the Hadamard product defined by $(a_{ij}) \star (b_{ij}) := (a_{ij} b_{ij})$. An explicit form of the dual basis is

$$h^*_n = (T^{-1}_{n,k})^* e_k \quad = \quad \sum_k T^{-1}_{k,n} e_k = \sum_{k|n} \sqrt{k}\mu(|n/k|)c_{n/k}(h_k)e_k.$$

4 Multi-channel, multi-sampling rate theorem

Let f be a member of the Paley-Wiener space PW_Ω defined as

$$PW_\Omega = \{f \in L^2(\mathbb{R}) : \operatorname{supp} f^\wedge \subseteq \Omega\}.$$

Then

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_\Omega f^\wedge(\omega) e^{i\omega t} d\omega.$$

In particular, for $n \in \mathbb{Z}$, and $\Omega = (-\pi/\tau, \pi/\tau)$ the sampled version of f is

$$f(n\tau) = \frac{1}{\sqrt{2\pi}} \int_\Omega f^\wedge(\omega) e^{i\omega n\tau} d\omega = \langle f^\wedge, e_n \rangle_{L^2(\Omega)}, \tag{12}$$

where τ represents the sampling period. The orthonormal expansion of $f^\wedge$ can be written

$$f^\wedge = \sum_n \langle f^\wedge, \overline{e_n} \rangle_{L^2(\Omega)} \overline{e_n}.$$

We apply (12) and find the discrete form of the Fourier transform, viz.

$$f^\wedge = f(n\tau)^\wedge = \sum_n f(n\tau)\overline{e_n} = \langle f(n\tau), e_n \rangle.$$

Similarly, the discrete form of a general transform can be written as

$$f(n\tau)^{\sim} = \sum_n f(n\tau)\overline{h_n}.$$

Our construction of the class of biorthonormal systems can now be employed to derive a new sampling theorem.

Theorem 4.1 *Let $\{h_n, h_n^*\}$ be a biorthonormal system such that $\{h_n\}$ satisfies the symmetry conditions (3) and (4), and the periodicity condition given by (5). Let $\{h_n\}$ be constructed such that $c_{km}(h_n)/c_k(h_n)$ is independent of k. Then*

$$f(n\tau)^{\wedge} = \sum_{m|n} c_m(h_1)\mu(|m|)f\left(\frac{n\tau}{m}\right)^{\sim}.$$

Proof We write $\langle f, e_n\rangle$ as

$$\begin{aligned} f(n\tau)^{\wedge} &= \left\langle (T^{-1})^* f, e_n \right\rangle = \left\langle \sum_{k|n} T^{-1}_{k,n} f(k\tau), h_n \right\rangle \\ &= \left\langle \sum_{k|n} f(k\tau)\sqrt{k}\mu(|n/k|)c_{n/k}(h_k), h_n \right\rangle. \end{aligned}$$

Put $n = mk$ and use $c_m(h_{n/m}) = \sqrt{m/n}c_m(h_1)$ to get

$$\begin{aligned} f(n\tau)^{\wedge} &= \left\langle \sum_{m|n} c_m(h_1)\mu(|m|)f\left(\frac{n\tau}{m}\right), h_n \right\rangle \\ &= \sum_{m|n} c_m(h_1)\mu(|m|)\sum_n f\left(\frac{n\tau}{m}\right)\overline{h_n}. \qquad \square \end{aligned}$$

This series can be interpreted as a parallel decomposition of the Fourier transform into an array of general transforms. It is an example of a multi-channel sampling theorem and it utilises the dilation property of the Fourier transform. In such a context, the m would denote the different channels. In each channel a general transform of the function $\{f(n\tau/m)\}_n$ is calculated. The results in each channel are scaled and summed to give $f^{\wedge}$. The final sum over m can be truncated to approximate $f^{\wedge}$ with arbitrary accuracy. The sampling rate decreases as m increases. Hence this sampling theorem is a multi-channel, multi-sampling rate, theorem.

5 Application to the Step transform

As a simple example of the sampling theorem given here, we consider bases which are formed from finite linear combinations of piece-wise constant functions. To this end, it is expedient to examine the cal and sal functions defined by

$$\begin{cases} \mathrm{cal}_\omega(t) = \operatorname{sgn}\cos\omega t \\ \mathrm{sal}_\omega(t) = \operatorname{sgn}\sin\omega t \end{cases}.$$

These are consistent with the definitions given by Elliot and Rao [4]. For $x \in \mathbb{R}$, the 'signum' function is defined as follows:

$$\operatorname{sgn} x = \begin{cases} x/\left|x\right|, & \text{if } |x| > 0 \\ 0, & \text{if } x = 0 \end{cases}.$$

The complex valued, two-level or binary, step function associated with the cal_ω and sal_ω functions is defined by

$$\psi_\omega(t) := \sqrt{\frac{\pi}{32}}\left(\mathrm{cal}_\omega(t) + i\,\mathrm{sal}_\omega(t)\right). \tag{13}$$

The corresponding step transform is given as

$$f^{\sim}(\omega) := \int_{\mathbb{R}} f(t)\overline{\psi_\omega(t)}dt. \tag{14}$$

This transform has the advantage of being straightforward to implement with a digital computer. Depending upon the interval in which a sample point, is taken at t_0, the function f at t_0 is added or subtracted. Indeed it is the convenient absence of multiplications which makes this transform appealing.

Commonly, the cal and sal transforms find applications in the area of sequency analysis. Here the working assumption is that the signal under question can be represented by a finite sum of square waves. This gives rise to the idea of Walsh functions [17] which are orthogonal bases with respect to sequency rather than to frequency. Elliot and Rao [4] give an extensive bibliography on this type of analysis. However, we will concentrate on the utility of $\{\psi_\omega\}$ in the context of frequency analysis and sampling theory.

The basis $\{\psi_\omega\}$ satisfies the symmetry conditions (3) and (4), as well as the periodicity condition (5). The Fourier series coefficients can be calculated as follows:

$$c_m(\psi_\omega) = \sqrt{\frac{\pi}{32}}(c_m(\mathrm{cal}_\omega) + ic_m(\mathrm{sal}_\omega)).$$

Now

$$\begin{aligned} c_m(\mathrm{cal}_\omega) &= \sqrt{\frac{\omega}{2\pi}} \int_{-\pi/\omega}^{\pi/\omega} \mathrm{cal}_\omega(t) e^{-im\omega t} dt, \\ &= \sqrt{\frac{\omega}{2\pi}} \left(-\int_{-\pi/\omega}^{-\pi/2\omega} + \int_{-\pi/2\omega}^{\pi/2\omega} - \int_{\pi/2\omega}^{\pi/\omega} \right) e^{-im\omega t} dt, \\ &= \begin{cases} \frac{2}{m\pi}(-1)^{\frac{m-1}{2}}\sqrt{2\pi/\omega}, & \text{for odd } m \\ 0, & \text{otherwise} \end{cases}, \end{aligned}$$

likewise

$$
\begin{aligned}
c_m(\mathrm{sal}_\omega) &= \sqrt{\frac{\omega}{2\pi}} \int_{-\pi/\omega}^{\pi/\omega} \mathrm{sal}_\omega(t) e^{-im\omega t} dt, \\
&= \sqrt{\frac{\omega}{2\pi}} \left(-\int_{-\pi/\omega}^{0} + \int_{0}^{\pi/\omega} \right) e^{-im\omega t} dt, \\
&= \begin{cases} -\frac{2i}{m\pi}\sqrt{2\pi/\omega}, & \text{for odd } m \\ 0, & \text{otherwise} \end{cases} .
\end{aligned}
$$

Together, the cal and sal Fourier coefficients yield

$$
c_m(\psi_\omega) = \begin{cases} m^{-1}\omega^{-1/2} & \text{for } m \in 4\mathbb{Z}+1 \\ 0 & \text{otherwise} \end{cases} . \tag{15}
$$

Consequently, the following corollary follows from Theorem 4.1.

Corollary 5.1 *Let $f^{\sim}$ denote the step transform, as defined in Equation (14). Then*

$$
f(n\tau)^{\wedge} = \sum_{\substack{m|n \\ m\in 4\mathbb{Z}+1}} \frac{\mu(m)}{m} f\left(\frac{n\tau}{m}\right)^{\sim}.
$$

References

[1] A. Aldroubi and K. Grochenig. Beurling-landau-type theorems for nonuniform sampling in shift invariant spaces. *J. Fourier Anal. Appl.*, 6(1):91–101, 2000.

[2] L. Bezuglaya and V. Katsnelson. The sampling theory for functions with limited multi-band spectrum I. *Anal. Anwendungen*, 12:511–534, 1993.

[3] P.L. Butzer, W. Splettsößer, and R.L. Stens. The sampling theory and linear prediction in signal analysis. *Jahresbericht der Deutschen Mathematiker Vereinigung*, 90:1–70, 1988.

[4] D.F. Elliot and K.R. Rao. *Fast transforms: algorithms, analyses, applications.* Academic Press, Orlando, 1982.

[5] S. Goldman. *Information theory.* Constable, London, 1953.

[6] J.R. Higgins. Sampling theory for Paley-Wiener spaces in the Riesz basis setting. *Proc. R. Irish Acad.*, 94A:219–236, 1994.

[7] J.R. Higgins. Sampling for multi-band signals. In: *An international conference on mathematical analysis and signal processing,* Cairo, pages 165–170. Contemporary Mathematical Series, AMS, 1994.

[8] J.R. Higgins. *Sampling theory in Fourier and signal analysis*. Clarendon Press, Oxford, 1996.

[9] D.L. Jagerman and L.J. Fogel. Some general aspects of the sampling theorem. *IEEE Trans. Inform. Theory*, 2:139–156, 1956.

[10] A Kohlenberg. Exact interpolation of band-limited functions. *J. Appl. Phys.*, 24:1432–1436, 1953.

[11] V.A. Kotel'nikov. On the carrying capacity of the ether and wire in telecomunications. Material for the first all-union conference on questions of communication. *Izd. Red. Upr. Svyazi RKKA*, 1933.

[12] D.A. Linden and N.M. Abramson. A discussion of sampling theorems. *Proc. IRE*, 47:1219–1226, 1959.

[13] P. Marziliano and M. Vetterli. Reconstruction of irregularly sampled discrete-time bandlimited signals with unknown sampling locations. *IEEE Trans. Sig. Proc.*, 48(12):3462–3471, 2000.

[14] A. Papoulis. Generalized sampling expansion. *IEEE Trans. Circuits and Systems*, 24:652–654, 1977.

[15] C.E. Shannon. Communication in the presence of noise. *Proc. IRE*, 37:10–21, 1949.

[16] I. Someya. *Waveform transmission*. Shukyo, Tokyo, 1949.

[17] J.L. Walsh. A closed set of normal orthogonal functions. *Amer. J. Math.*, 45:5–24, 1923.

[18] J.M. Whittaker. On the functions which are represented by the expansions of the interpolation theory. *Proc. R. Soc. Edinb.*, 35:181–194, 1915.

[19] P.M. Woodward. *Probability and information theory*. Pergamon Press, Oxford, 1953.