Mathematical Models and Reality - a Constructivist Perspective

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September 14, 2007

Abstract: To explore the relation between mathematical models and reality, four different levels of reality are distinguished: observer-independent reality (to which there is no direct access), personal reality, social reality and mathematical/formal reality. The concepts of personal and social reality are strongly inspired by constructivist ideas. Mathematical reality is social as well, but constructed as an autonomous system in order to make absolute agreement possible. The essential problem of mathematical modelling is that within mathematics there is agreement about "truth", but the assignment of mathematics to informal reality is not formally analyzable, and it is dependent on social and personal construction processes. On these levels, absolute agreement cannot be expected.

Starting from this point of view, repercussion of mathematical on social and personal reality, the historical development of mathematical modelling, and the role, use and interpretation of mathematical models in scientific practice are discussed.

1 Introduction

In this paper I give an account of the relation between reality and mathematical models as I see it. My definition of mathematical models is quite general. Tentatively, I call a collection of mathematical objects a **mathematical model** whenever all objects of the collection have an interpretation in terms of real objects, and the results of mathematical operations performed on the objects have such an interpretation as well. It should be well defined which mathematical operations can be meaningfully carried out on the objects. To make the definition more precise at this point would open a Pandora's box, particularly because I do not want to define "real objects" in the Introduction. Note, however, that I include logic as far as it is formalized by a calculus as "mathematical objects" in my definition.

The relation between mathematical models and reality lies at the bottom of all scientific reasoning involving mathematics. Therefore, how this relation is viewed has strong implications concerning the role of science in our perception of the world and the interpretation of scientific results. In one way or another, it has been discussed by many philosophers, Aristotle, Kant, Tarski, Wittgenstein, Popper, just to mention a few. The purpose of the present paper is to explain my own view. It is not my aim to discuss the conceptions of other philosophers, though I am well aware that much of the ideas laid out in this paper have been developed by other thinkers in the past and some have been around since a very long time. As a statistician and not a philosopher by education, I am also quite sure that there are relevant references which I am not aware of. My main philosophical influences are constructivist writers like Ernst von Glasersfeld, Heinz von Foerster, Humberto Maturana and Kenneth Gergen. Some references will be given in the notes.

I call my own perspective "constructivist" because the idea of constructing realities is central to it. I use some conceptions of the constructivist writers mentioned above, but I do not want to imply that I agree (or disagree) with them (or some of them) on questions not explicitly discussed here. Actually, as far as my understanding of constructivism goes (at least radical constructivism), every constructivist constructs his or her own ideas of "constructivism" anyway, and it should not be surprising that not all of "us" come up with the same ideas. Note that philosophical constructivism has nothing to do with what is usually referred to as "constructivism" in the philosophy of mathematics (Troelstra, 1991, see also note 38)).

I would like to present my ideas starting from the very basics, assuming nothing, but unfortunately this is impossible. The impact of communication and the use of language on our perception of reality is crucial, and therefore the fact that I have to assume a certain understanding of the words I use already in the beginning is an inevitable obstacle. Our construction of reality necessarily involves circularity and feedback loops[1)] so that we can never start at zero once we have been involved in communication and have acquired language[2)]. Furthermore, an account of the relation between mathematical models and non-mathematical reality can obviously not be given in a purely formal, mathematical way, so that I cannot start with mathematical axioms and precise definitions. Precise definitions in informal language cannot be given because they would necessarily make reference to terms which are not already precisely defined[3)].

For similar reasons, I cannot prove my ideas to be true, nor do I think that they can be refuted. Much of these ideas is about what we cannot know for sure, and how we can proceed despite that. To some extent it is always a matter of choice how we see the world and what meaning we attach to our perceptions. Therefore, the present conception is an offer to think about, and perhaps adopt, a certain point of view, but it is not an ideology of which I have any proof that it "is true" or "has to be adopted"[4)].

The text has two different levels. The main body of the text without the notes is meant to present the basic ideas in condensed form so that it is easily possible to get an overview. I am, however, well aware that readers who are not familiar with the kind of thinking applied here will find some of the ideas difficult to understand and will strongly disagree with some others. Therefore I introduced a lot of notes, which not only give references, but also contain some additional comments, examples and clarifications. I consider the notes as an essential part of the text.

In Section 2, the basic conception of the role of mathematical models in science and its relation to different realities is presented, after observer-independent, personal and social reality have been introduced. Section 3 analyzes mathematical modelling from a historical perspective, which helps to understand why some mathematical models seem "natural" while others are controversially discussed. Section 4 discusses implications of the presented approach for scientific practice. In a further paper, I plan to apply the ideas presented here to the discussion about the foundations of probability.

The use of the first person singular in some parts of this paper does not mean that I claim the credit for inventing these ideas. It is meant in a rather modest way to say that I hold these ideas, others may as well, but I do not want to assume that every reader agrees with them automatically or has to agree with them, which seems to me the implication of the usual use of "we" in scientific and philosophic publications. See note 7) for an explanation of my use of "we".

Notes

1) Before we learn what a "precise definition" of a term is and become aware of its use, our communication is obviously not based on such definitions, so that the concept of precision itself cannot be defined precisely. But we can go back and try to become more and more precise about the definitions of the terms we have already used before, so that the idea of possible precision arises out of imprecision by feedback.

2) Constructivism has been criticized quite often for denying that we have to assume some common ground to avoid solipsism. I think that this is a misconception. Certainly we have to assume some common ground to be able to communicate, and therefore to live. But we do not have to do that because it could be proven objectively in any way what this common ground has to be. It is perfectly compatible with constructivism to accept that *some* common ground has evolved through the practice of living and communication, and that we could neither develop nor communicate our ideas without making reference to it. But we do not have to attach more authority to it than just that.

3) Other authors (e.g., Apostel, 1961, Casti, 1992) have tried to formalize their theory of the relation of mathematical modelling to reality, but such approaches can be seen as formal models in itself and the analysis of the relationship of these theories to informal reality may be seen as asking for another (meta-)theory of formal modelling. Similar comments apply to the representational theory of measurement (Krantz et al., 1971), which formalizes the relation of quantification to reality, and to mathematical model theory (Manzano, 1999).

4) A further major criticism of constructivist and generally relativist ideas is that if there is no objective truth, it is meaningless to say that constructivist or relativist statements are true (and contradictory if constructivists do so). This criticism ignores that "truth" can no longer have the same meaning from a constructivist point of view. As constructivists do not think that the objectivist conception of truth makes sense, they do not (or at least should not) claim that constructivism is objectively true, but this does not make constructivism any weaker, except in the eyes of objectivists who insist that their own construct of "truth" needs to be applied. See 2.5 and 2.7 for constructivistically valid conceptions of "truth".

2 Domains of reality

2.1. I have access to the reality outside myself only through my perception. My perception is actively constructed by my brain, by which I mean that all impressions from the outside are processed by it[5)]. Therefore,

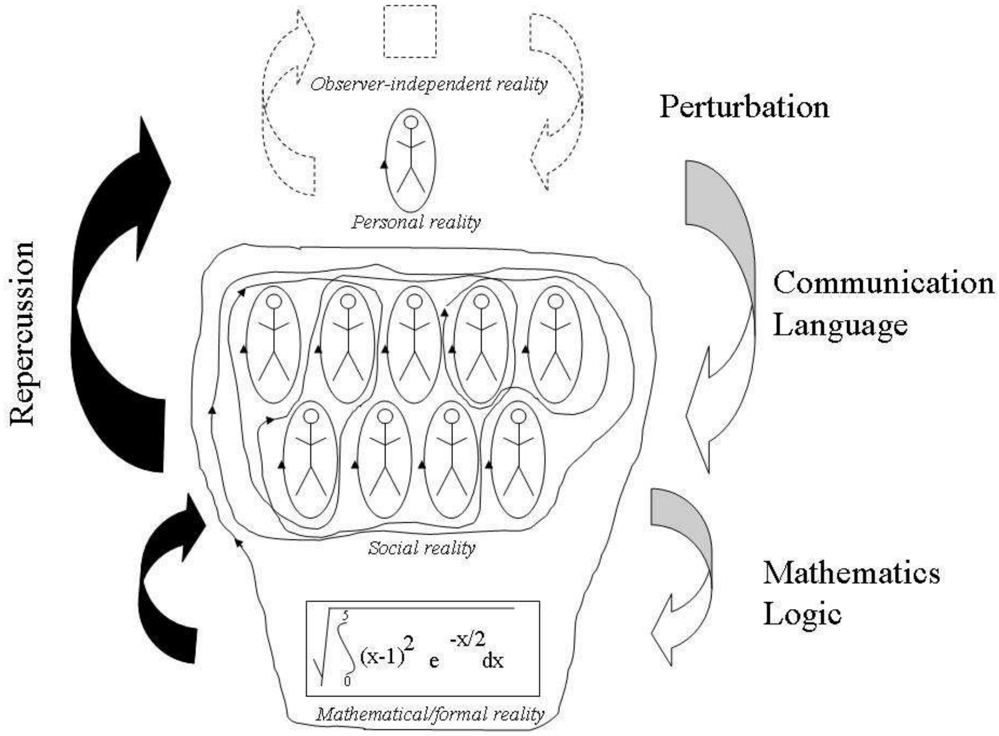

Figure 1: Illustration of four domains of reality, namely observer-independent reality, personal, social and formal/mathematical reality, as outlined in Section 2. For esthetic reasons I have omitted science as a wider social system containing formal reality. Mathematical/formal reality can be seen as closed system, but is part of social reality as well.
Circles with arrows show systems that define their own border (social reality can be seen as comprising several such subsystems, which may be overlapping).
Neither personal nor social reality has direct access to the observer-independent reality. Therefore I have drawn it with a dotted line (realists could use a solid line, add science and connect it with the observer-independent reality if they think that science has a more direct approach to it than other social systems and individuals). Note, however, that the access of individuals to other individuals and social reality is only via constructing them as part of the observer-independent reality outside themselves, so that it could be said that from the point of view of any particular personal (or social) reality, *all* other realities should be dotted. The difference is that the observer-independent reality does, for lack of observers, not qualify as a "point of view".

I cannot know how the reality outside is in an unprocessed state, independently of my acts of perception. I do not have access to any objective reality, "objective" here meaning "independent of the observer", because there is no other means to check the "objectivity" of observations of an observer than making reference to observations of other observers[6]. I will use the term "**observer-independent reality**" in the following for such a reality outside, which at least can be said to exist as an idea shared by most of us, whether (and how) or not it exists objectively.

Notes

5) The construction processes, however, are usually unconscious, even though it is conceivable to increase the access to it. Thus, usually, I cannot choose how my perception is constructed.

6) This can be seen as the core idea of radical constructivism, see, e.g., von Förster (1984), von Glasersfeld (1995)

2.2. I distinguish observer-independent reality from the "**personal reality**" of an individual. There is a different personal reality for every individual. Our personal reality is the reality experienced by us[7]. It comprises our sensual perceptions, our thoughts and our conceptions about the world[8]. We have direct access to our personal reality[9], but not to the observer-independent reality. We cannot know what the observer-independent reality looks like. We cannot know whether there is a unique observer-independent reality for all individuals, nor even whether the observer-independent reality exists "objectively" at all. We can know, however, whether the idea of a unique observer-independent reality (or an observer-independent reality of which at least some aspects are unique) is part of our personal reality[10].

Notes

7) The use of "we" in connection with personal realities here means that it is part of *my* personal reality that I define my concepts in a way that these statements hold for all human beings, which relies on the assumption that my concept of human beings allows these general definitions.

8) For example, if I see a tree, I go away and come back later and see something very similar, I could, as part of my personal reality, assume that

this is the same tree, which has also been there in the meantime.

9) The definition of a personal reality raises a difficulty with the subconscious. I take personal reality to mean only what we experience consciously, but this may (and does, in my case) include the conception that there are subconscious aspects of our experience. This means that if I am convinced that a subconscious component is involved in my experiences, this conviction is part of my personal reality, and I may even experience consciously, but afterwards, such subconscious components of experience. For example, as a sceptical thinker, I could have the idea that many material things that appear in my perception are actually deceptions, i.e., perceptions which are inappropriate representations of the observer-independent reality, and that I cannot trust them. But I may be convinced by thinking retrospectively about my actions, that I acted confidently *as if* my perceptions are to be trusted, and that therefore, subconsciously, I have relied much stronger on my perceptions of the observer-independent reality than I had been consciously aware of.

10) The well-known debate about quantum physics and its interpretation should illustrate clearly why the idea of a "unique observer-independent reality" cannot be taken for granted.

2.3. The precise state of our personal reality cannot be communicated. We can never know whether the words that we use and that we believe to understand correspond to the same experiences of other human beings with whom we communicate[11)]. Therefore I further distinguish "**social reality**" from "personal reality" and "observer-independent reality". Social reality comprises all acts of communication. An image of it appears in the personal realities of the individuals by the interpretation of the perceived acts of communication, which is part of personal reality, but distinct from social reality[12)]. Likewise, images of personal realities and psychological states, but not the personal realities itself, appear in communication.

Social systems can be defined by distinguishing possibly overlapping groups of communicators or particular modes of communication. It is possible to distinguish social realities belonging to different social systems, but these distinctions are usually not clear-cut[13)].

Notes

11) The same applies to expressions of body language and the like.

12) This amounts to a distinction very similar to the one made by Luhmann (1995) between the psychological and social self-organized systems. According to Luhmann, self-organized systems are defined by operations constructing the border of the corresponding system (self-organizing systems have been defined to be operationally closed by Maturana and Varela, 1980). The psychological system is defined by an operation different from communication which defines the social system. Both systems are therefore constructed to be distinct and part of each other's environment, which corresponds to my view of social and personal reality.

13) Luhmann (1995) has a stronger concept to distinguish self-organizing social systems, see note 12), which allows to define more clearly separated social realities. However, it seems to me that his definition is too restrictive and does not capture some structures of which to speak as "social systems with their own social realities" makes sense. For example, his definition of an economical social system by the money circuit is convincing to me, but I do not see how the true/false code, which he uses as a defining operation for the scientific system, yields an operational closure.

2.4. There is repercussion between personal and social reality. A major part of social reality is generated by attempts of individuals to communicate their personal perceptions[14)]. On the other hand, social reality is not only perceived by individuals. It further has a strong impact on the construction of personal reality, because people use language[15)] to think. Perceptions are connected, structured and even changed by language[16)].

I call patterns of perceptions or actions that are perceived as belonging together "**constructs**"[17)]. Constructs can be personal and social[18)]. Personal constructs refer to patterns of personal perceptions, social constructs refer to patterns of communicative acts. Constructs are often referred to by words, and the same word usually refers to a social construct and personal constructs of many persons[19)].

The meaning of constructs does neither have to be precisely defined[20)] nor consistent[21)].

Notes

14) I do not think, though, that exchanging personal perceptions is the principal aim of communication. I rather think that communication can be seen primarily as a technique of self-organization to deal with perturbations (i.e., perceptions of events affecting the individual, see Maturana and Varela, 1980). or to fulfill needs, for which it turned out that the attempt to exchange personal perceptions is often a useful tool.

15) "Language" can be understood in a general sense here, including for example artistic and mimic expressions.

16) It is possible to get an impression of how perception is changed by language by noticing how our, say, visual perception focuses on those observations for which we have words. Automatically we rather see a wooden table than all the small patterns on the table for which we do not have descriptions. We can deliberately concentrate on seeing details for which we do not have words, but this is not what we usually do (assuming, of course, that what holds for me in this respect holds for most if not all of the readers). Furthermore, we do not only perceive of the table what is directly visible, but we also see it as a "table", which is man-made, to be used as a table, which almost certainly has a leg more that we cannot see from where we are because otherwise it could not stand etc. Thus, we add some ideas that we learned through language.

17) A more sophisticated description of constructs, with which the present conception should be compatible, is given by von Förster (1984). It can be roughly summarized as "fixed (stable) points of recursive coordinations of actions".

18) Radical constructivism (von Glasersfeld, 1995) and social constructionism (Gergen, 2000) can be seen as mainly distinguished by focusing strongly on the personal, the social level of construction, respectively. I see both of them as essentially important. Constructivism should not be reduced to one of them.

19) Obviously constructs of different individuals and social systems are connected by the use of the same word, but that does not mean that the underlying perceptions are identical. Furthermore, the fact that patterns of communicative acts are perceived by individuals may cause confusion. However, the perception of communicative acts referred to by the word "fear" in social interaction can be properly distinguished from the direct personal perception of fear. Note that some people use the word "social construct" to emphasize that these constructs do *not* correspond to "real objects" by which they often mean "objects of the observer-independent

reality" but which can be interpreted, from the viewpoint taken here, as patterns of their personal perception. Some examples are discussed in Hacking (1999). I rather think that constructs are positively existent (by means of construction, in the reality in which they are constructed), and that this does not imply any negative statements about "real existence", referring to observer-independent or any other domain of reality.

20) The development of language, be it in the history of mankind, be it of a growing child, starts from actions and collections of examples rather than general definitions. As long as language is not formalized, even if a seemingly general definition is given, its elements can be traced back to initially imprecise concepts, and it remains unclear whether it is appropriate outside the domain of perception of the individual or social system that adopts the definition.

21) Even the personal constructs of a single individual are not necessarily consistent because the person may have contradictory thoughts.

2.5. We perceive that many constructs, for which we have words, are very stable[22]. More precisely, there are personal constructs that are consistent with different sensual perceptions at different times from different points of view. Furthermore, we observe that the corresponding social constructs (i.e., the ones to which the same word refers in communication) are stable as well and that other people behave consistently with the communicative acts and our perception concerning these constructs.

This can be taken as evidence for the belief that the constructs either are "representations" of some items in the observer-independent reality or even "direct perceptions" of such items. In my account, however, I prefer to remain agnostic about these interpretations[23]. The construction of the personal reality of a child growing up can be perceived as an inner process triggered by actions and reactions to events going on in its environment in order to survive in a manner as pleasant as possible. The strong dependence on its parents and a quest for control and reliability of the own reality[24] can account for the emergence of stable personal constructions corresponding to the communication and behaviour of its social environment[25] (of which the parents are the most prominent members) without the necessity to assume that there is a unique objective basis for these constructs[26].

This means that the concept of "truth" cannot refer to observer-independent reality in oder to make operational sense. However, it is still possible that individuals have consistent (possibly flexible) concepts of "personal truths" (and lies) and that social systems have (more or less) stable concepts of "truth" referring to communicative acts[27)].

It also means that the idea that people "understand" each other or "agree" has to be interpreted as "they can be observed to behave as if their constructs match"[28)].

Notes

22) As examples, I mainly have simple material things like tables in mind here, but this is not meant to be restrictive.

23) As long as it does not raise problems (i.e., instabilities in the personal reality or disagreements in communication), it is possible to operate with stable constructions in a straightforward way that does not have to deviate in any way from a "naive" realist's behaviour, and therefore we do not have to worry about whether the constructions represent something "objectively real". If there are disagreements and instabilities, however, it seems to me to be much more fruitful to allow the term "reality" (personal and/or social) for all existing points of view, and to attempt to resolve the problem by negotiations and actions *without prescribing that the solution has to be a unique and consistent.* "Agreement to disagree" or even persistent inconsistencies can be tolerated as long as the involved individuals feel that they can get on with their lives in an acceptable way.

24) The terms "as pleasant as possible" and "quest for control and reliability" are meant to describe a motivation behind actions leading to stable constructs that seems conceivable to me. They are neither meant to be normative, nor exhaustive of the class of possible motivations. As we do not perceive other people's motivations directly, generalization does not seem to be justified. I only hold that motivations are easily conceivable that can explain stable constructs reasonably.

25) This holds for adults as well, in principle, though this is less relevant because most of the strongly stable constructs are built in childhood.

26) I emphasize again that I do not deny such a unique objective basis either. The point is that it cannot be observed and therefore the issue cannot be decided.

I also emphasize that the claims I am making here are not in itself meant as references to observer-independent reality. I am well aware that I describe my personal perceptions and the sense I make of them (I see a potential agreement about them, though). Particularly I am well aware that the claim that social reality including parents "really exists" is not stronger than to claim the existence of the observer-independent reality.

On the other hand, the claim that personal reality exist is stronger, because denying it would by definition mean to deny that we perceive anything (including observer-independent reality and social reality). However, it would be a fallacy to conclude from this that the personal reality has general priority (as radical constructivism sometimes seems to suggest). Communication (social reality) and the existence of a material environment (observer-independent reality, including other people) are crucial in order to construct the personal reality (without knowing how these realities look like *objectively*), so that I rather see a circular relationships between these three.

27) The presumably most stable concept of "truth" exists within mathematical/formal reality, see 2.7.

28) "Behaving" includes "communicating" here. The constructs may "match" in different ways depending on whether it is about "understanding" or "agreement". Obviously, it depends on the observer whether "understanding" or "agreement" is ascribed to a situation.

2.6. The main (defining) objective of **science** (interpreted as a social system), as I see it, is to establish an (ideally) growing body of stable constructs about which general agreement is possible[29)30)31)32)33)]. This requires a communicative process in which different personal realities are somehow synchronized[34)]. Part of this process is that a language has to be created which is defined as precisely as possible[35)], and which emphasizes the agreements between different personal realities. Furthermore, general agreement requires that the individuals adapt their personal realities to scientific language and scientific ways of observation. They may discard or re-interpret perceptions and thoughts that are incompatible with the scientific world-view. Therefore, the scientific quest for general agreement affects the personal realities of those who take part or are exposed to it[36)].

Notes

29) The term "possible general agreement" is obviously imprecise. By "possible" I mean that scientists aim at constructs of which they believe

that everybody *who understands enough of the subject* agrees with them. "Agreement" means that the person who agrees decides deliberately that the scientific construct is consistent with her personal reality, which, however, can only be checked by communication, in social reality, subject to the difficulties to communicate the personal perceptions properly. But how can it be known whether general agreement is "possible" in the case that agreement is not already general (i.e., in the usual situation)? I do not want to specify this. Several rules and methods for scientific inquiry and communication have been introduced (e.g., transparency, reproducibility, statistical tests, the peer review process; some of which may be changed or even abandoned over time) that have proved to be helpful in order to make existing scientific results understandable and acceptable to (more or less) independent thinkers. My point of view is that these rules do not define science in itself but are justified only as long and as far as they serve the primary aim of general agreement. The term "who understands enough of the subject" is problematic, because it gives scientists the possibility to discard disagreeing world-views and to restrict attempted agreement to specialists. This leads sometimes to a quite authoritative and narrow-minded practice to communicate science, in spite of aiming at "deliberate agreement". Potentially fruitful points of view may be suppressed because they seem to be too threatening to a broad consensus within the scientific community. It is also possible that different social systems arrive at incompatible sets of ideas by methods that could legitimately claim to be "scientific". This is a reflection of the fact that aiming at a growing body of stable generally agreed upon constructs is extremely ambitious and some disagreement will always be met. Therefore, science has to be open to some extent (in order to be consistent with the aim of general agreement) but restrictive to some extent as well, in order to generate some progress. It is unclear if there could be a "right balance" and there is certainly some unpredictable self-organization at work; I would interpret Kuhn's (1962) view so that "normal science" works rather restrictively but openness is necessary to let the elements of "paradigm shifts" grow, which are needed if "normal science" is perceived to be "in trouble". The problem is that it can only be seen *post hoc* which of the ideas seen initially as incompatible lead to later generally accepted paradigms.

30) Most principles of "good science" serve this aim more or less directly, for example transparency, replicability of experiments, openness to criticism and discussion. I have the impression that Feyerabend (1993) is right in that no general definition of "the scientific method" can be given.

At least, the attempts to do so up to now led to much less stable and agreed upon constructs than the often quite anarchist work of the scientists itself.

31) A more traditional idea about the main objective of science is "to find out truths about observer-independent reality". Of course, if it is assumed that there is a unique observer-independent reality which is somehow accessible, its truths should manifest themselves in stable personal constructs which agree among people. In fact, there is no other means to find out about these truths than to consider the personal realities and whether and to which extent social agreement about the personal observations and ideas exists. This implies that my conception of science does not directly disagree with the traditional one. It only remains agnostic about the traditional assumption of a unique accessible observer-independent reality. However, my conception seems to be more supportive of an "agreement to disagree". If it is not an end in itself to find a unique objective truth, a perfectly valid scientific agreement could be that "the following views exist (. . .) and we do not have the means to decide scientifically between them." Of course, this could only be called a general agreement as long as nobody insists that the issue has to be decided.

32) The given description of the main aim of science is meant to be general enough to cover the pragmatic aspects of science ("stability" can mean that reliable predictions are enabled), but not be restricted to it (stable constructs do not have to be of immediate practical use). I think that the motivation behind of much scientific work is pragmatic, but this does not distinguish science from many other activities, while I tried describe what makes science special.

33) The definition given here is mainly meant to be descriptive. It should be normative only in a definitory sense, i.e., "if it does not support general agreement, then I would not call it science". But I do not imply ethical value here. Science, as I see it, is not a value in itself. In some circumstances, scientific agreement and unification may not be perceived as useful or "good". Ethical judgements about the value of scientific thinking have to be based on other sources than science alone, presumably in a case-wise manner.

34) The realist interpretation of this would be that, as long as personal realities are kept "objective", finding out the truth about the observer-independent reality automatically synchronizes personal realities. This is not problematic in situations where people feel that their personal

realities are consistent with scientific results, but I think that it does not give a very helpful account of situations in which disagreements arise.

35) As has already been said in the Introduction, definitions can never be fully precise because initially language starts from imprecise terms. However, it can be found out (and hopefully agreed upon) how agreement and understanding can be improved, namely, for example, by using operational and "directly observable material" terms (though the precise meaning of the terms "directly observable" and "material" may be prone to disagreement).

36) It is the standard realist view that the occurrence of (more or less) general agreement is very strong evidence in favour of the existence of a unique accessible observer-independent reality. This argument becomes much weaker if agreement is considered as the product of an interactive process aiming at agreement, which involves changing personal realities, i.e., observations and world-views. Nevertheless, I hope that even a realist can agree that it is possible and reasonable to describe science as such an interactive process.

2.7. Mathematics[37)] in its recent formalized form[38)] can be regarded as a closed[39)] social system generating its own reality, "**mathematical** (or formal) **reality**". The claim of mathematics is to provide a communicative domain in which absolute agreement is possible, and constructs are absolutely stable, because the mathematical objects and operations are well defined and abstract, i.e., cleaned of individual connotations (non-communicable links to personal reality). Within formal mathematics (and logic), "true" and "false" are well defined concepts referring to operations within formal reality.

Note that whether the claim of possible absolute agreement is "really" fulfilled can only be decided by informal communication (in social reality) and personal perception (in personal reality). Different opinions may exist[40)].

Notes

37) What I write here basically applies to formal logic as well. I do not consider the question of major importance here whether formal logic is a part of mathematics or just works in an analogous way, and I therefore do not discuss it further. Particularly it makes sense to speak of "formal

logical models of arguments" in the same way as of "mathematical models of (parts of) reality".

38) This refers to a Hilbert-type formalist interpretation of mathematics, though it does not need completeness. It is clear that this interpretation does not comprise everything that legitimately can be called "mathematics". It makes sense to me, as a *philosophical* constructivist, to grant mathematical objects existence *as constructs within the formalist mathematical reality* if they have a stable operational meaning within formal mathematics. This does not require their *explicit* construction - which distinguishes me from *mathematical* constructivism (Troelstra, 1991). I am not claiming that formalism is the "correct" philosophy of mathematics, only that its view of mathematics seems to work best to define the kind of "mathematical reality" I have in mind in the present approach. It should not be forgotten, however, that formal mathematics emerged from concepts that are linked much more directly to personal experience. A brief historical account is given in Section 3.

39) By "closed" I mean here that mathematics can be seen as a formal system the rules of which clearly define what "inside" and "outside formal mathematics" is. Strictly spoken, the closure cannot be complete, because informal language has to be used at least to make an initial definition and to explain how axioms can be operated with, but I regard the closure as "about as complete as a subsystem of social reality can be".

40) My experience is that in mathematics people are either able to attain agreement or regard *themselves* as incompetent. This indicates that the current development of mathematical formalism at least does a very good job in supporting as absolute as possible an agreement among the people who feel entitled to take part in the mathematical discourse. Note that I distinguish questions inside mathematical reality like how to derive implications from axioms and definitions from questions like which axioms and definitions are reasonable, which I locate outside mathematical reality.

2.8. It should be obvious from 2.6 and 2.7 that the development of formal mathematics fulfills an essential scientific aim. However, as long as science is concerned with non-abstract phenomena appearing in the personal realities and/or informal social reality, mathematics can only be useful if mathematical objects are assigned to non-abstract constructs. I call this "**mathematical modelling**". The most prominent use of mathematical modelling is to generate propositions about non-abstract constructs by interpreting true mathematical results in terms of the constructs, which qualify for general agreement because they are formally "true" in mathematical reality.

The basic problem of mathematical modelling is that the assignment of formal mathematical objects to non-abstract constructs cannot in itself be formally analyzed[41]. Non-abstract constructs are, by virtue of being non-abstract, essentially different from mathematical objects. Furthermore, it is inherent in the process of abstraction that some qualities of the constructs to be abstracted have to be cleared[42], which means that the content of the mathematical result can never be the same as the content of its interpretation in terms of non-abstract constructs. Formal "truth" can never apply to the assignment. This requires informal personal decisions and social negotiations about whether and to what extent the interpretation can be accepted.

As mathematical objects, in mathematical modelling, are associated with personal and social constructs, people may be stimulated to think and communicate about these constructs explicitly or implicitly in terms of the corresponding mathematical objects[43], which means that mathematical reality reacts on and changes social and personal realities[44].

Notes

41) It has actually been tried to formalize the process of the assignment of formal mathematical objects to non-abstract constructs to some extent, see e.g., Krantz et al. (1971), Casti (1992). But then this formalism becomes a formal model in itself, and to further formalize its correspondence to its underlying non-abstract constructs leads to infinite regress.

42) The qualities to be cleared are at least those which cannot be communicated in scientific terms and those about which there is disagreement among the people who are meant to agree about the mathematical model. Because the domain of social and personal realities is much richer than that of mathematical reality, in most cases many further features are cleared as well.

43) For example, some people may *identify* the "amount of intelligence" of a man with his IQ value, and others, who are careful enough to prevent that, may still talk about intelligence in a way that implies that the intelligence of people can be ordered on a one-dimensional scale, clearing at

least temporarily the perception that there are a lot of inherently multidimensional or non-measurable constructs of intelligence around.

44) By analogy to note 36), this gives an explanation of how stable and general constructs referring to mathematical models can be without making reference to the observer-independent reality, which realists would see as the major source of stability. (Of course, this does not disprove realism.)

3 The development of mathematical modelling

3.1. The constructivist approach outlined above implies that it is difficult to give a precise description of a concept[45), as long as it is not well defined within a formal system. The concept may appear in different personal realities and it may, in social reality, be controversial and inconsistent.

In the present section, I try to give a brief description how I see the process that led to the present construction of formal mathematics (as I made reference to in 2.7 above) and its role in modelling reality. As a constructivist, I do not assume that there is any unique and "objectively true" meaning of mathematics or mathematical objects, which is what, to my impression, much of the philosophy of mathematics is after. Instead, a constructivist approach to describe what a concept such as mathematics "really is" (in social reality) would describe as precisely as possible the process that led to its construction and the present and past operations and ideas involving it, taking into account that such a description is just one out of many possible narratives of this process. I ignore the "as precisely as possible"-wording here and I only give a sketch. Apart from the fact that I am not an expert on the history of mathematics[46)], it has to be kept in mind that the number of sound preserved sources for mathematics before the Greeks is very small and the knowledge of the beginnings of mathematics, which according to evidence originated before writing, can only be limited.

Notes

45) I think that it is difficult under any philosophical approach, but most non-constructivism ones are better at hiding the difficulties.

46) My knowledge stems from books such as Burton (2007) and Kropp (1994).

3.2. Counting can be seen as the origin of mathematical concepts. Objects like grazing sheep, and goods to be traded, have been counted by notches on bones and by fingers even before the beginning of writing. The principle of mathematical abstraction, as described in 2.8 above, has already been present at this stage, and it could be considered as an instance of mathematical modelling: the fingers or notches are assigned to non-abstract constructs such as the sheep, ignoring individual differences between the sheep (as well as the fingers or notches) which would be much more difficult to communicate, and about which agreement seems to be much more difficult. In trading, this enables a person to send a servant to organize a particular quantity of goods, implicitly or explicitly assuming that they are sufficiently similar, without having to negotiate personally about the value dependent on the precise individual conditions.

Another early indication of the use of what we today would consider as mathematical objects (but not mathematical modelling, though it could have been inspired by geometric thinking connected to problem solving in "real life") are geometric ornaments in prehistoric art.

At this early stage there is no evidence that mathematical objects have been considered as entities in their own right, let alone as making up a consistent closed system.

3.3. From this, more abstract mathematical objects such as numbers and geometrical forms emerged. It is not known how long it took until these objects have been perceived to have an existence detached from the concrete material constructs the dealing of which they supported. There are Babylonian, Chinese and Egyptian sources dealing with numbers without making direct reference to what is counted, though the practical relevance of all the given computations is immediate. There is evidence in these writings that people were aware of the particular stability with which arithmetical techniques could be taught and applied, and of the strong potential for agreement, leading to religious interpretations (in ancient China) and claims that they give *"insight into all that exists, knowledge of all obscure secrets"*[47)], even though from today's view the techniques may seem quite modest. I presume that the quest for the authority coming from the stability and generality of possible agreement was, at least from some point, a driving force in the development of mathematics, although this can only be speculation[48].

Notes

47) This is from p. 37 of Burton (2007), citing the Egyptian Rhind papyrus from about 1650 B.C

48) There is no evidence of any controversial discussion (or proof) of mathematical results before the Greeks, so that strong authority was ascribed also to results that are wrong from our present view. I interpret this as an indication that it could make sense to say that the idea of absolute certainty of the mathematical results was rather actively constructed than passively observed (though this is presumably only conceivable if the results have been successful, i.e., enough precise, in applications again and again).

3.4. Before the Greeks, mathematics had still been tightly connected to the practice of living. The Greeks went much further. They introduced the idea of proof (going presumably back to Thales), general theorems using letters for general numbers (the Pythagoreans) and eventually a closed theory starting from axioms (Euclid), which made it possible to develop mathematics regardless of non-abstract constructs. The Greeks became explicitly aware of the difference between an abstract mathematical object and the material objects to which the mathematical theorems were applied[49]. Nevertheless, the Greek ideas of the observed material reality and mathematical objects were still strongly linked. Starting from the Pythagoreans and later in Platon's philosophy, the abstract entities were seen as the universal, more authoritative reality, of which the material reality gives only an imperfect idea[50]. They perceived mathematics as beautiful, stable and useful, and their idea that nature obeys an essentially mathematical order is highly influential up to the present day[51]. This stage can be interpreted as the beginning of a closed, formal reality made up by mathematics, operating on itself. However, the use of mathematics was still restricted to the fields from which it had been developed, and it was closely linked to an intuition stemming from these fields - it took more than 2000 years before it was discovered that mathematics as a formal system provides space to develop alternative ideas like non-Euclidean geometry.

Notes

49) *"We know from Aristotle that Protagoras (...) used against the geometers the argument that no such straight lines and circles as they assume exist in nature, and that (e.g.) a material circle does not in actual fact touch a ruler at one point only."* (Heath, 1981, p. 179)

50) Note that such ideas can hardly be appropriately discussed from a modern scientific "true"/"false"-perspective, while making reference to their social construct of the "observer-independent reality" and its relation to personal perceptions seems to me to do them much more justice.

51) *"The Pythagorean discovery belongs to the strongest driving forces of human science (...) If mathematical structure can be recognized as the essence (*Wesenskern*) of musical harmony, the reasonable order of the nature around us has to have its source in the mathematical laws of nature."* (Heisenberg 1958 **I translated this from a German source - have to check it!**)

3.5. The Greeks begun from practice and arrived at abstraction. "Modern mathematical modelling" took the opposite direction. Galilei started to use pre-existing formal mathematics to think about observational constructs such as gravity, which were remote from the origins of the mathematical objects. Galilei's *Discorsi* (1638) start from mathematical definitions of uniform and uniformly accelerated movements, not from experiments, and proceeded deductively. Galilei produced a lot of mathematically deduced physical results which he himself did not confirm by experiments (Koyré, 1978), and he was seemingly aware of the abstract nature of his physics[52)]. Some of the assumptions were obviously unrealistic, at least at his time (e.g., vacuum conditions). Guided by a theory like this, experimenters (and engineers who wanted to make use of the theory) had to become concerned about actively producing ideal conditions under which the assumptions hold. To me it seems to be obvious at this point at the latest that mathematical models were not only observed in nature (or developed from natural observations), but mathematical thinking also changed nature directly, through the way we perceive nature. The idea that precise (mathematical) physics proceeds from theoretical assumptions and nature has to be forced (if possible at all) to deliver the conditions under which the results hold can be found for example in Newton's work and in Kant's philosophy[53)].

The results of mathematical physics were identified by most people and even by most scientists with objective results about the observer-independent reality, at least up to the debates about the meaning of quantum physics, but the awareness rose that the connection between mathematics and reality is more problematic. The non-Euclidean geometry and other counter-intuitive mathematical constructs stimulated a philosophical debate about the nature of mathematics. Obviously, it was now possible to construct mathematical entities that were not connected to any observable reality anymore.

Notes

52) In a letter, in 1637, Galilei wrote: *"If experience shows that properties as those that we have deduced are confirmed by the free fall of bodies in nature, we can claim without danger of error that the concrete movement is identical to the one we have defined and assumed. If this is not the case, our proofs, which only hold under our assumptions, do not lose anything of their power and consistence"* (translated by myself from a German quote in Ortlieb, 2000) **No idea how to find this in English**.

53) See Ortlieb (2000) for quotes.

3.6. The term "model", to my knowledge, was coined in science by Hertz (1894), who for the first time distinguished a mathematical model and the modelled reality explicitly, and saw the necessity to discuss the appropriateness of every particular model.

The development cumulated in the formalist philosophy of David Hilbert. He had the aim to give mathematics a solid foundation without making any reference to the observed reality, in order to apply an as independent and as elaborate as possible version of what I call "closed mathematical reality" to a very wide range of topics, not restricted to the traditional bastions of mathematical modelling[54)][55)].

Notes

54) Hilbert (1900) wrote: *"Let us turn to the question from what sources this science derives its problems. Surely the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena. (...)*

But, in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its *independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortunate ways, new and fruitful problems, and appears then itself as the real questioner. (...)*

In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. And it seems to me that the numerous and surprising analogies and that apparently prearranged harmony which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science, have their origin in this ever-recurring interplay between thought and experience." This description has a lot in common with my view, though Hilbert (like most of his successors) does not take into account the mind- and communication-dependence of "experience" and seems rather to imply that experience is objective and comes directly from the observer-independent reality.

55) Gödel's incompleteness theorems limited the success of Hilbert's mathematical program, but as far as I see, it does neither have negative implications about the idea of mathematics as autonomous formal system, nor about the applicability of mathematics.

3.7. In the 19th and, much stronger, 20th century, mathematical thinking and modelling appeared in more and more disciplines, e.g., social science, medicine, biology, and psychology[56)]. However, most of these uses of mathematics have been discussed much more controversial than in the more traditionally formalized disciplines, particularly physics[57)]. The historical development could be instructive. Numbers, geometrical forms, partitions and the measurement of lengths, weights and time lie at the origin of mathematical thinking. The oldest mathematical objects have emerged out of human activity concerned with these constructs, and mathematical development was based on the corresponding intuitions for thousands of years. This means that the connection of mathematics with these concepts is extremely well established and stable and it is hardly ever put into question (which does not mean that *identification* is justified). If we apply mathematical models, it could be said that we tend to think about the modelled reality "in terms of the original concepts" or using them as metaphor[58]. Obviously, this seems to be the more problematic, the clearer we are aware of the differences between the construct to be modelled and the initial mathematical intuition.

Many of these discussions have never been resolved, and there is a variety of incompatible points of view, all of which are reasonably consistent in itself, for example about the mathematical modelling of probability or intelligence and the interpretation of these models. History shows how mathematics developed into a system that is very much independent of the material, observational constructs of personal and social reality, even though it was initiated by basic human activity. It can be seen how come that some connections between mathematical entities and social/personal constructs seem very clear and "natural" to most of us[59], while others are so controversial.

It seems to me that the conception of the domains of reality, considering formal mathematics as a domain on its own, taking into account different personal realities, being agnostic about whether unification based on the structure of the observer-independent reality would be possible *in principle*, and emphasizing the role of social reality and the quest for agreement as a driving force of science, can give a fruitful description of this state of affairs.

Notes

56) The case of economics is somewhat peculiar, because economics is based on money, which can be seen as a "materialized" mathematical model of economic value with an older history than formalized mathematics. Economic theory (including mathematical economics) helped to uncover how problematic this original "model" is.

57) See for example the chapter about "The Struggle to Extend a Calculus of Probabilities to the Social Sciences" in Stigler (1986).

58) For example, measurement of temperature by a thermometer required to think about temperature in terms of a length. The usual intuition about probabilities has to do with partitions ("1 in 20"). One could reduce initial concepts further and say that we think about time in terms of lengths if employing real numbers to measure time periods.

59) A connection can be drawn between the historical development and

the way mathematical knowledge is acquired nowadays. Usually, most of the initial, oldest concepts are learned in a quite active, explorative way by children first, while the need to teach more modern concepts in a limited amount of time leads to an acceleration of speed in the teaching of mathematics (additionally to the fact that these concepts had several thousand years less to establish themselves), and the resulting intuition for these concepts is usually much less stable and more superficial.

4 Mathematical modelling in scientific practice

4.1. According to the view described in the previous sections, mathematical models belong to scientific communication. Mathematical objects are assigned to (personally or socially) real constructs. This enables precise and well-defined communication and the derivation of "true" implications. Mathematics is constructed in order to enable potentially absolute agreement about the "truth" of such statements. However, this is only meaningful *within mathematics*. The assignment of the (personally or socially) non-abstract constructs to formal objects is not itself accessible by formal analysis[60]. Since mathematical and non-abstract constructs do not belong to the same domain of reality, they cannot be identified. Therefore, it is not of much help to say that "the mathematical statements can be interpreted as true statements about non-abstract reality[61] *if the mathematical assumptions hold*". The mathematical assumptions are abstract, and to say that the "hold" in the non-abstract social and personal domain implicitly assumes that the formal and the non-abstract domains can be identified, which again is inaccessible to the formal concept of "truth". Therefore, it is more appropriate to say that mathematical modelling is about the investigation of the implications of ways of thinking about reality.

Notes

60) More precisely, this is only possible to the price of causing new problems of the same kind, see note 41).

61) Those who make such statements usually mean observer-independent

reality here, but according to the conception of the present paper, it makes rather sense to think about social and personal reality.

4.2. How should mathematical models be chosen, given that (formal) "truth" cannot be attributed to the choice of the model (at least not if it is not embedded in a formal super-model)? Within the general aim of supporting scientific agreement, there are several conceivable purposes of mathematical modelling, and of course a model has to be assessed by whether it is fit for its purpose.

- It is possible to **improve mutual understanding** by developing a mathematical model that models the point of view to be communicated. Mutual understanding is a kind of agreement, not about the truth or validity, but about the content of a statement.
- Mathematical modelling can support **agreement** about the modelled reality, as long as the model and its interpretation can be accepted by everyone involved. This requires communication about the potentially different personal points of view and decisions about which aspects of the reality should be modelled and which are unimportant and can be ignored. Ignorance of some aspects of the modelled realities is always necessary, because abstraction is essentially about removing personal connotations and details that hinder unified understanding and agreement. Individual perception and potentially relevant communication such as literature can be, and usually are, extremely complex, and the abstract model enables to make the decisions transparent about what are conceived to be the crucial aspects. Communication is also required about formalized ways to observe, i.e., **measurements**. Measurement is guided by models, and therefore modelling makes it possible to check and reproduce scientific results.
- Mathematical modelling **reduces complexity** and can make clearer and simpler perception of the reality possible.
- Often models are used for **prediction**. Note, however, that I do not interpret predictive models as "approximations of observer-independent reality". Prediction is therefore about the implications of a way of thinking about the reality. Usually, in order to use models for prediction, it is assumed that crucial conditions

remain constant or at least that their rate of change remains constant, which often differs from how we perceive the non-abstract constructs involved. Of course, the quality of predictions can be assessed by making observations in the future. However, in some setups, particularly in economics (e.g., stock markets) and the social sciences, actions are based on mathematical predictions and these actions have an effect on the future, which is usually not modelled. Therefore, the prediction quality of the model cannot be assessed properly.

A surprising finding is that in many setups, when deriving predictions statistically from existing data, flexible "black box"-prediction machines without a straightforward interpretation do a better job than "realistic" models[62)]. This illustrates that good prediction is essentially different from finding an agreed upon model of reality.

- Models can provide **decision support** by generating comparable consequences from models formalizing different decisions. (The remarks about prediction above apply again.)
- Models can be used to **explore different scenarios**, for example optimistic and pessimistic ones in climate change research, which can give us a quantifiable idea of uncertainty. Note that here, again, it is not necessary a reasonable strategy to look for the "most realistic" model.
- Mathematical models often have surprising implications and give us a new, different view of the modelled phenomena. This can **stimulate creativity**[63)].
- It may be a major benefit of mathematical models to **guide observations** by highlighting *disagreement* between observational data and model predictions, or between personal and modelled constructs. I have seen several cases in statistical consultation in which the most valuable discoveries came from the inspection of outliers and data patterns that were incompatible with the initial models. It can actually be quite valuable if a model turns out to be unrealistic.
- Sometimes mathematical models are perceived to be **beautiful and elegant**, which I see as a perfectly legitimate purpose of modelling as long as beauty and elegance are not used as argu-

ments to convince people of the "truth" of the model[64].

As shown, different purposes require different kinds of discussion about how a model relates to the underlying personal and social realities, but in more or less all cases it is instructive to discuss these relations. Many of the benefits of modelling come from exploring *differences* between a model and personal and social constructs and observations, so that the present point of view emphasizes the exploration of such differences strongly. A particular benefit of this could be the idea of mutual understanding of differences of views, the agreement to disagree. The present conception highlights that mutual understanding can never be taken for granted, not even in science. The meaning of words and concepts cannot be assumed to be unique among individuals. Disagreements cannot be settled by just referring to "objective truth" but need negotiation. To uncover such disagreements is essential for science, as I see it.

Notes

62) See, e.g., Breiman (2001).

63) Note that many technological achievements have been stimulated by mathematical models, but their final form is usually not a straightforward conversion of a formal construct, but is modified strongly in practice. It is rather the creative potential of modelling than "good approximation of reality" that matters here.

64) There is a very long tradition of associating mathematics with esthetics, see 3.2.

4.3. Traditionally, differences between model and perceived reality would normally either lead to making the model more complex by modelling some of the missing details, or to regrets that "the model should be more complex but it is not possible, because that would make analysis too cumbersome". Note, however, that modelling further details does not just mean that the model becomes "more realistic", but also that more potentially problematic assignments of formal objects to non-abstract constructs have to be made. Depending on the purpose (which, according to the present conception, cannot just be "fitting objective reality"), this may be useful or not. Therefore, in many cases it becomes acceptable and positively justifiable to make the model not as complex as possible, and to ignore some (personally or socially) real details.

4.4. The present approach implies that the mathematical correctness of derived results within a model does not work as a sufficient argument in favour of the "truth" of the result interpreted in terms of the modelled constructs. According to my experience, mathematical modelling is often used to make results appear more authoritative (sometimes without serving any further aim, and sometimes without a proper discussion about the relation of the model to the modelled reality). This would not be possible if people were more aware that **the essential problem of mathematical modelling is informal**, namely the connection of the mathematical objects and the non-abstract constructs, for which there is no formalization[65)].

Notes

65) Even if people agree upon the assignment of mathematical objects to non-abstract entities, they do not necessarily have to agree upon the interpretation of mathematical results. It may happen that characteristics of the modelled entities seem to be unimportant initially and are ignored in the model, but later they become important for particular interpretations of mathematical results. An illustration for this is classical mechanics, which before the appearance of relativity theory and quantum mechanics had been agreed upon and interpreted so generally that it seemed to be a serious problem that its results did not "hold" on the micro and macro level. However, this did not mean that the model had to be dropped, but rather that the domain for which the interpretation of the results yielded satisfactory predictive power had to be restricted.

4.5. A concluding aspect of the approach presented here is that it takes the repercussion of mathematical modelling on social and personal realities explicitly into account. Mathematical models change our thinking, and this makes it favorable to discuss not only how our pre-existing realities are reflected by the model, but also what kind of thinking is implied by modelling, what kind of changes to our realities may be stimulated, and whether this is desired[66)]. A sentence like "the model represents (fits/approximates) the reality very well" cannot only be read as a statement about the model, but also about the personal reality of the person who makes the statement. "Approximation" can work both ways, thinking can be adapted to the model, and differences between the model and the individual perception may vanish because the perception of differences may be reduced[67)].

The clarity and stability of mathematics comes to the price of abstraction and distance from personal and social perceptions. The benefits of formalization and agreement always have to be weighed against the dangers of reduction and unification.

Notes

66) An example is the growing influence of league tables comparing schools and universities quantitatively. This corresponds with the introduction of more and more unified assessments with more far-reaching consequences. But instead of just measuring the quality of schools passively, the whole procedure has strong effects on the perception of schools and also on teaching and learning. It increases the focus on assessments strongly, which many regard as counter-productive.

67) This happens rather if the model is advertised as objective and authoritative, instead of being open and positive about its limitations.

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