Spectral Sequences

Jamil Nadim
Department of Mathematics, University College London
e-mail: a.jamil@ucl.ac.uk
Supervisor: F E A Johnson

0. Spectral Sequences

Usually, we begin with some initial data (like a filtration $\mathcal{F}$ of a topological space X) and just like a matrix is an array of numbers, a spectral sequence is an "infinite" book if you like, a series of pages, each of which is a $2-D$ array of abelian groups. On each page there are maps called differentials that "go to the left" between the groups which form chain complexes. The homology groups of these chain complexes are precisely the groups which appear on the next page, taking the "homology of the homology" recursively eventually (under suitable conditions) leads to each group stabilizing to the E^∞ page of the book, from which we interpolate our desired (co)homology groups (again under certain conditions). The general mantra of spectral sequences is that we'd like to have terms at the E^1 or E^2 pages that we somehow understand and wind up with the E^∞ page that relates to something we'd like to compute (although it is possible to run the knowledge the other way and deduce things about the E^1s from the E^∞s we wind up with, the recovery is often complicated and is usually a collection of extension problems).

0.1. Spectral Sequence Definition.

Formally, an E^r spectral sequence (of homological type) is a collection $\{E^r_{p,q}, d^r_{p,q}\}^{r\geq 1}_{p,q\in\mathbb{Z}}$ of bigraded abelian groups $E^r_{p,q}$ and differentials (i.e. homomorphisms) $d^r_{p,q}$ for each $E^r_{p,q}$ such that:

(1) $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$ of bidegree $(-r, r-1)$, where the total degree $p+q$ on the bigraded abelian group decreases by 1 "like homology" for maps moving r units leftwards and $r-1$ units upwards simultaneously such that:

(2) $d^r_{p,q} \circ d^r_{p+r,q-r+1} = 0$ for any same page composition

$$\cdots \to E^r_{p+r,q-r+1} \xrightarrow{d^r_{p+r,q-r+1}} E^r_{p,q} \xrightarrow{d^r_{p,q}} E^r_{p-r,q+r-1} \to \cdots$$

and isomorphisms of $E^{r+1}_{p,q} \cong H_*(E^r_{p,q})$ given by:

(3) $E^{r+1}_{p,q} = \frac{Ker(d^r_{p,q})}{Im(d^r_{p+r,q-r+1})} \cong H_*(E^r_{p,q})$ which appear on the next page.

Definition. *An E^k spectral sequence $\{E^r_{p,q}, d^r_{p,q}\}^{r\geq k}$ is called first quadrant if and only if $E^k_{p,q} \neq 0 \Rightarrow p \geq 0$ and $q \geq 0$, that is, the lattice point (p,q) belongs to the first quadrant of the plane and if this holds true for k, it clearly holds for all $r \geq k$.*

- Addendum 1: The E^0 page is simply the filtration and its quotients taken on the chain complex, the E^1 page, where our story begins, is made up of the standard horizontal homology long exact sequences of the E^0 page. We usually begin at the E^2 page as this is where the differentials begin mapping in interesting directions (i.e. possible non-zero maps between groups on different levels), the elements of $E^2_{p,q}$ are a second-order approximation of the homology of $E^1_{p,q}$, and so the E^{r+1} is the bigraded homology group of the preceding (E^r, d^r). Note for later convenience, we may consider spectral sequences beginning at any stage E^k for $k \in \mathbb{N}$ where $r \geq k$.

- Addendum 2: The E^r and d^r determine E^{r+1} but not necessarily d^{r+1}, finding d^{r+1} in terms of what has come before is much more complicated, unless certain conditions are included, for example, considering cohomology spectral sequences which are naturally endowed with a multiplicative structure on their differentials due to their cup product. Also note that an object on the lattice point $E^{r+1}_{p,q}$ is a subquotient of $E^r_{p,q}$ due to (3).
- Addendum 3: We shall only encounter first quadrant spectral sequences, so all other quadrants consist of zero lattices throughout the rest of this essay. This condition implies that for fixed p and q at a particular lattice point, as one keeps turning successive pages for $r >> 1$ whilst focusing on (p, q), the d^r differentials will eventually go to zero (i.e. have trivial codomain) making them zero maps which do not go through (p, q), so they affect groups at the lower left hand of the quadrant for a limited time only. Thus each d^r eventually becomes trivial. So for large enough r and fixed (p, q), $E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}$ i.e. things settle down and the groups $E^r_{p,q}$ become fixed for all r passed a certain point.

Dual Definition. *A spectral sequence of cohomological type starting at stage E_k, is a sequence $\{E^{p,q}_r, d^{p,q}_r\}_{r\geq k\geq 1}$ of bigraded objects together with differentials $d^{p,q}_r$ going "to the right":*

$$d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$$

of bidegree $(r, -r+1)$ satisfying $d^{p+r,q-r+1}_r \cdot d^{p,q}_r = 0$ and such that the homology of E_r is E_{r+1}, i.e. $H_*(E_r) \cong E^*_{r+1}$.

Explanatory Remark. *In other words, it is the same thing as a homology spectral sequence, reindexed via $E^{p,q}_r = E^r_{-p,-q}$, so that $d^{p,q}_r$ increases the total degree $p+q$ of $E^{p,q}_r$ by 1. Furthermore, the cohomology version has extra structure, namely $H^*(X)$ has a ring structure induced by the cup product, and the differentials $d^{p,q}_r$ respect this ring structure: $d(\alpha \cdot \beta) = d(\alpha) \cdot \beta + (-1)^{p+q} \alpha \cdot d(\beta)$. More precisely, there are bilinear products $E^{p,q}_r \times E^{s,t}_r \to E^{p+s,q+t}_r$ which are just the standard cup product when $r = 2$.*

1. Leray-Serre Spectral Sequence

1.1. **The Leray-Serre Theorem.** Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a fibre bundle with fibre map $p : E \to B$, under certain conditions, namely:

- The base space B is pathwise and simply connected
- the fibre F is pathwise connected
- and $\pi_1(B)$ acts trivially on $H_*(F; G)$ (i.e. trivial monodromy on the homology of the fibres)

Then there is a first quadrant spectral sequence $(E^r_{p,q}, d^r_{p,q})$ with:

(1) $d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$
(2) $E^{r+1}_{p,q} = \frac{Ker(d^r_{p,q})}{Im(d^r_{p+r,q-r+1})}$
(3) $E^2_{p,q} \cong H_p(B; H_q(F; G)) \quad \rightsquigarrow H_{p+q}(E)$
(4) $E^\infty_{p,q} \cong \frac{H_{p,q}}{H_{p-1,q+1}}$

The last condition gives the E^∞-page, where the spectral sequence stabilizes and the limiting groups are denoted $E^\infty_{p,q}$, which are the groups we are after. The punchline is that with sufficient assumptions on our fibration, the groups $E^\infty_{p,q}$, which we constructed from the data $H_p(B; H_q(F; G))$, are related to the groups $H_{p+q}(E)$. Furthermore, if there is monodromy, the spectral sequence still works by replacing $E^2_{p,q} = H_p(B; H_q(F))$ with $E^2_{p,q} = H_p(B; \mathcal{H}_q(F))$, where $\mathcal{H}_q(F)$ is a local system on B, but we will not go into that.

Fibre Bundle Relationship. The fundamental group $\pi_1(B)$ acts on the fibre F (i.e. it acts on all the algebraic invariants of F). So if $h : F \to h(F)$ is an algebraic invariant, then $h(F)$ is a module over the group ring $\mathbb{Z}[\pi_1(B)]$.

Suppose you are given a fibre bundle:

$$\begin{tikzcd} F \arrow[dr, "i"] & \\ & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

Construct pullbacks of coverings and universal covering spaces:

$$\begin{tikzcd} F \arrow[dr, "i_*"] & & F \arrow[dr, "i"] & \\ & \hat{E} \arrow[rr, "Proj_1"] \arrow[d, "p_*"] & & E \arrow[d, "p"] \\ & \tilde{B} \arrow[rr, "Proj_2"] & & B \end{tikzcd}$$

where $\tilde{E}$ and $\tilde{B}$ are universal covers of E and B respectively, with standard definition: $\tilde{E} = \{(\tilde{b}, e) \in \tilde{B} \times E \ : \ Proj_2(\tilde{b}) = p(e)\}$ with

$\tilde{B}$ having a similar definition. Since $\pi_1(B)$ acts on $\tilde{B}$, therefore it also acts on $\tilde{E}$. If $g \in \pi_1(B)$, then $g : b \to g \cdot b \in \tilde{B}$.

$$\therefore \quad \begin{tikzcd} p^{-1}(b) \arrow[rr, "\cong"] \arrow[dr] & & p^{-1}(g \cdot b) \arrow[dl, "identification"] \\ & p^{-1}(b) & \end{tikzcd}$$

Hence, $g : h(p^{-1}(b)) \xrightarrow{\cong} h(p^{-1}(b))$. An easy hypothesis is to assume $\pi_1(B) = \{0\}$ so that the fundamental group acts trivially on F.

Specializing. Take a fibre bundle of CW-complexes $F \to X \xrightarrow{\pi} B$ (where we call X the total space now) such that B is simply connected. Then consider the skeletal filtration of the base space B:

$$B^0 \subset B^1 \subset \cdots \subset B^{p-1} \subset B^p \subset \cdots \subset B$$

where $B^p/B^{p-1} = \bigvee S^n$ is a wedge of n-spheres by the classification of bundles. The p-skeletons of B induce a filtration on X:

$$\pi^{-1}(B^0) \subset \pi^{-1}(B^1) \subset \cdots \subset \pi^{-1}(B^{p-1}) \subset \pi^{-1}(B^p) \subset \cdots \subset \pi^{-1}(B)$$

such that $\pi^{-1}(B^p)/\pi^{-1}(B^{p-1}) = \bigvee S^n \times F$. By setting $X^i := \pi^{-1}(B^i)$, we get the same filtration on X:

$$X^0 \subset X^1 \subset \cdots \subset X^{p-1} \subset X^p \subset \cdots \subset X$$

We are essentially filtering the cellular chain complex $C_*(X)$ by letting $F_p(C_*(X)) = C_*(\pi^{-1}(B^p))$. Then $F_pC_*(X)/F_{p-1}(C_*(X))$ is the quotient complex $C_*(\pi^{-1}(B^p))/C_*(\pi^{-1}(B^{p-1})) \cong C_*(\pi^{-1}(B^p), \pi^{-1}(B^{p-1}))$. By excision, the homology of this complex is the direct sum:

$$\bigoplus_{e^p} H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p))$$

over the p-cells e^p of B. But since e^p is contractible, the fibre over it is trivial and so homotopy equivalent to $e^p \times F$. So

$$H_*(\pi^{-1}(e^p), \pi^{-1}(\partial e^p)) \cong H_*(e^p \times F, \partial e^p \times F) \cong H_*(D^p \times F, S^{p-1} \times F)$$

which by the Künneth theorem is just $H_{*-p}(F)$, which we can also interpret as $H_p(D^p, S^{p-1}; H_{*-p}(F))$. But since $H_*(S^n)$ is torsion free, we can define the Künneth theorem as $H_n(S^p \times F) \cong H_p(S^p) \otimes H_{n-p}(F)$. The inclusions $\pi^{-1}(B^p) = X^p \hookrightarrow X$ induce maps $H_n(X^p) \hookrightarrow H_n(X)$ and so we set:

$$F_p^n := Im(H_n(\pi^{-1}(B^p)) \to \pi^{-1}(B)) = Im(H_n(X^p) \to H_n(X))$$

This gives a filtration where p is the cellular dimension, n the homological dimension

$$\mathcal{F} = (0 \subset F_0^n \subset F_1^n \subset \cdots \subset F_{p-1}^n \subset F_p^n \subset \cdots \subset F_n^n = H_n(X))$$

Then the theorem of Leray-Serre says:

$$\boxed{E^{\infty}_{p,n-p} \cong F^n_p / F^n_{p-1}}$$

which we get by doing the E^1 spectral sequence of the filtration $\mathcal{F}$:

$$E^1_{p,q} = \bigoplus_{e^p} H_p(D^p, S^{p-1}; H_q(F)) \cong H_{p+q}(F_p/F_{p-1}) \Rightarrow_p H_{p+q}(X)$$

The map d^1 takes this to $\bigoplus_{e^{p-1}} H_{p-1}(D^{p-1}, S^{p-2}; H_q(F))$ by the boundary map of the long exact sequence of the triple (B^p, B^{p-1}, B^{p-2}). By the Künneth Theorem and the classification of bundles over S^p and B^p:

$$H_{p+q}(F_p/F_{p-1}) \cong C_p(B) \otimes H_q(F) \quad \cdots (\spadesuit)$$

By taking the homology of $(\spadesuit)$, we get the desired powerful result:

$$\boxed{E^2_{p,q} = H_p(B; H_q(F)) \Rightarrow H_{p+q}(X)}$$

Naturality. Thus we have shown for maps of fibrations:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & B \arrow[d, "\phi"] \\ F' \arrow[r] & E' \arrow[r] & B' \end{tikzcd}$$

such that the fibre of $b \in B$ maps to the fibre over $\phi(b) \in B'$, we get maps $E^r_{p,q} \to E'^r_{p,q}$ (including the case $r = \infty$) which respect all differentials and isomorphisms.

Example. An easy application occurs when most of the $E^2_{p,q}$ terms vanish. Consider the fibre bundle X over an n-sphere with fibres m-spheres. We seek to compute the homology $H_\bullet(X)$,

$$\begin{tikzcd} S^m \arrow[r] & X \arrow[d, "p"] \\ & S^n \end{tikzcd}$$

where we know that $H_i(S^r; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, r \\ 0 & \text{otherwise.} \end{cases}$ We use the Leray-Serre Theorem in a "direct" fashion to fill up the E^2 page as follows:

$$E^2_{p,q} = H_p(B; H_q(F)) = H_p(S^n; H_q(S^m; \mathbb{Z})) = \begin{cases} H_p(S^n; \mathbb{Z}) & \text{if } q = 0, m \\ 0 & \text{otherwise} \end{cases}$$

Then the only non-trivial $E^2_{p,q}$ groups will be $\mathbb{Z}$s at $p = 0$ and $p = n$. Thus there are four lattice points: $(p, q) = (0, 0), (0, m), (n, 0), (n, m)$. We get the following E^2 page with one possible transgressional differential $d^r_{n-r,r-1}$ (edge homomorphism) if $(0, m) = (n - r, r - 1)$ for some r:

$$\begin{tikzcd}
m & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \\
\vdots & 0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & 0 & 0 & 0 & 0 & \mathbb{Z} \arrow[uuuulllll, dashed, "d^r_{n-r,r-1}"'] \\
 & 0 & 1 & 2 & 3 & \dots. & n
\end{tikzcd}$$

All differentials are clearly trivial if the above differential does not exist for any r. So we compute the groups on the E^3 page using:

$$E^{r+1}_{p,q} = \frac{Ker(d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1})}{Im(d^r_{p+r,q-r+1} : E^r_{p+r,q-r+1} \to E^r_{p,q})}$$

but the d^r are all trivial so the spectral sequence collapses, all groups stabilize giving:

$$E^2_{p,q} \cong E^3_{p,q} \cong \cdots \cong E^\infty_{p,q}$$

By Leray-Serre we know: $E^2_{p,q} = H_p(S^n; H_q(S^m;\mathbb{Z})) \Rightarrow H_{p+q}(X)$ so:

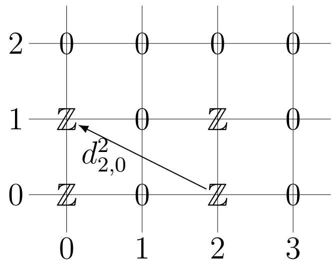
$$\boxed{H_\bullet(X) \cong \bigoplus_{p+q=\bullet} E^\infty_{p,q}}$$

$$\therefore \quad H_\bullet(X) = \begin{cases} \mathbb{Z} & \text{if } \bullet = 0, m, n, m+n \\ \text{OR if } m = n & \\ \mathbb{Z} & \text{if } \bullet = 0, 2n \\ \mathbb{Z}^2 & \text{if } \bullet = n \end{cases}$$

Special Case. If $(0,m) = (n-r, r-1)$ for some r, then there are other possibilities as the differential $d^r_{n-r,r-1}$ connects non-trivial groups on different levels which the relative homology does not record as explained earlier. Take for example the Hopf fibration of S^3 with S^2 as a subquotient:

$$\begin{tikzcd}
S^1 \arrow[r] & S^3 \arrow[d, "p"] \\
 & S^2
\end{tikzcd}$$

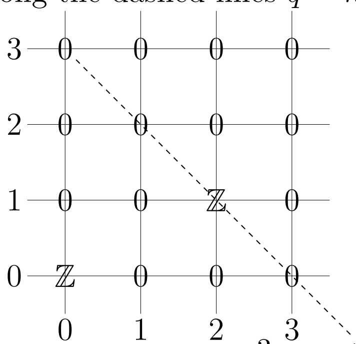
By Leray-Serre: $E^2_{p,q} = H_p(S^2; H_q(S^1;\mathbb{Z})) \Rightarrow H_{p+q}(S^3;\mathbb{Z})$ which has only one possible non-zero differential $d^2_{2,0}$ on the E^2 page:



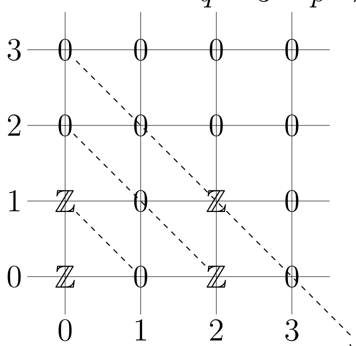
Since we know $H_*(S^3;\mathbb{Z})$ already, we must have $d^2_{2,0}$ is an isomorphism so $d^2_{2,0} = \pm 1$, otherwise for $S^2 \times S^1$, we must have $d^2_{2,0} = 0$. We get the following E^3 pages by taking the homology of the E^2 page using

$$E^3_{p,q} = \frac{Ker(d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1})}{Im(d^2_{p+2,q-1} : E^r_{p+2,q-1} \to E^2_{p,q})}$$

to get the $E^3_{p,q}$ groups which remain stable (the differentials move up by atleast three rows now) and are therefore isomorphic to the E^∞ groups, so to "read-off" the n^{th} homology group, we sum the groups along the dashed lines $q = n - p$ for all n.



$q = 3 - p \Rightarrow H_3(S^3) = \mathbb{Z}$



$q = n - p \Rightarrow H_n(S^2 \times S^1)$

$$\therefore \ \underline{H_n(S^2 \times S^1;\mathbb{Z})} = \begin{cases} \mathbb{Z} & \text{if } n = 0,1,2,3 \\ 0 & \text{otherwise.} \end{cases}$$

2. Spectral Sequence Calculations

2.1. **The Complex Projective Plane $\mathbb{C}P(-)$.** The computation of $H_*(\mathbb{C}P(-))$ involves "working backwards" as we already know the homology of the total space (the E^∞ page) and using this fact, we can compute the homology of the base space (the E^2 page). We give two examples, the first is a "finite case" $H_*(\mathbb{C}P(2))$.

Example 1. Consider the fibre bundle:

$$\begin{tikzcd} S^1 \arrow[r] & S^5 \arrow[d, "\pi"] \\ & \mathbb{C}P(2) \end{tikzcd}$$

We know that $H_i(S^r;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ \mathbb{Z} & \text{if } i = r \\ 0 & \text{otherwise.} \end{cases}$

We know the E^∞ page will give the homology of S^5, so we will have $\mathbb{Z}$s on $E^\infty_{0,0}$ ...$(\star)$ and $E^\infty_{4,1}$ only.

Using Leray-Serre and the definition of a spectral sequence:

$$E^2_{p,q} = H_p(B;H_q(F;G)) \Rightarrow H_{p+q}(E)$$
$$E^2_{p,q} = H_p(\mathbb{C}P(2);H_q(S^1;\mathbb{Z})) \Rightarrow H_{p+q}(S^5)$$

(i) $d^2_{p,q} : E^2_{p,q} \to E^2_{p-2,q+1}$
(ii) $E^3_{p,q} = Ker(d^2_{p,q})/Im(d^2_{p+2,q-1})$
(iii) Using $(\star)$ or by connectedness of $\mathbb{C}P(2)$; $E^2_{0,0} = E^2_{0,1} = \mathbb{Z}$ since $E^2_{0,1} = H_0(\mathbb{C}P(2);H_1(S^1;\mathbb{Z})) = H_0(\mathbb{C}P(2);\mathbb{Z}) = \mathbb{Z}$.
(iv) $E^2_{p,q} = H_p(\mathbb{C}P(2);H_q(S^1;\mathbb{Z})) = \begin{cases} H_p(\mathbb{C}P(2);\mathbb{Z}) & \text{if } q = 0,1 \\ 0 & \text{if } q \geq 2 \end{cases}$

So on the E^2 page, the only possible non-zero terms are on the p-axis and the "line" $q = 1$, thus $E^2_{p,q} = H_p(\mathbb{C}P(2);H_q(S^1;\mathbb{Z})) = 0$ for $q \geq 2$ as $H_p(\mathbb{C}P(2);0)$ will have zero coefficients.

$q \backslash p$	-1	0	1	2	3	4	
2		0	0	0	0	0	E^2 page
1	0	$\mathbb{Z}$	$H_1(\mathbb{C}P(2))$	$E^2_{2,1}$	$E^2_{3,1}$	$E^2_{4,1}$	(So far!)
0		$\mathbb{Z}$	$H_1(\mathbb{C}P(2))$	$E^2_{2,0}$	$E^2_{3,0}$	$E^2_{4,0}$	

$d^2_{1,0}$

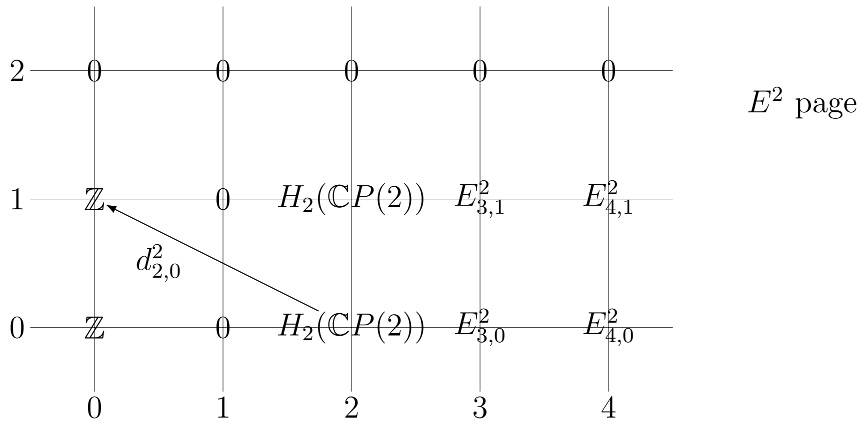
Step 1. Compute $H_1(\mathbb{C}P(2);\mathbb{Z}) = E^2_{1,0} = E^2_{1,1} = ?$

$$E^3_{1,0} = \frac{Ker(d^2_{1,0} : E^2_{1,0} \to E^2_{-1,1})}{Im(d^2_{3,-1} : E^2_{3,-1} \to E^2_{1,0})} = \frac{Ker(d^2_{1,0} : E^2_{1,0} \to 0)}{Im(d^2_{3,-1} : 0 \to E^2_{1,0})} = \frac{E^2_{1,0}}{0} \cong E^2_{1,0}$$

But since all the subsequent differentials that go to (or from) $E^3_{1,0}$ will come from a zero (or go to a zero) are trivial, the group will stabilize thus:

$$E^2_{1,0} = E^3_{1,0} = E^4_{1,0} = \cdots = E^\infty_{1,0} = 0$$

as $E^\infty_{1,0}$ is a subquotient of $H_1(S^5;\mathbb{Z}) = 0$. $\therefore$ $E^2_{1,0} = H_1(\mathbb{C}P(2);\mathbb{Z}) = 0$.



Step 2. Show $H_2(\mathbb{C}P(2);\mathbb{Z}) \cong \mathbb{Z}$. To see this it suffices to show that $d^2_{2,0} : E^2_{2,0} \xrightarrow{\cong} E^2_{0,1}$ is an isomorphism, for then $H_2(\mathbb{C}P(2);\mathbb{Z}) = E^2_{2,0} = E_{0,1} = \mathbb{Z}$ (known). Hence we must show:

(a) $d^2_{2,0}$ is injective

(b) $d^2_{2,0}$ is surjective

Injectivity of $d^2_{2,0}$:

$$E^3_{2,0} = \frac{Ker(d^2_{2,0} : E^2_{2,0} \to E^2_{0,1})}{Im(d^2_{4,-1} : 0 \to E^2_{2,0})} = \frac{Ker(d^2_{2,0})}{0} = Ker(d^2_{2,0})$$

Now $E^r_{2,0} = E^r_{2,0}$ for $r \geq 3$ as all $d^r = 0$ for $r \geq 3$. But since $H_2(S^5;\mathbb{Z}) = 0$, it follows that $E^3_{2,0} = \cdots = E^\infty_{2,0} = 0$ as $E^\infty_{2,0}$ is a subquotient of $H_2(S^5;\mathbb{Z}) = 0$. $\therefore$ $Ker(d^2_{2,0}) = 0 \Rightarrow d^2_{2,0}$ is injective.

Surjectivity of $d^2_{2,0}$:

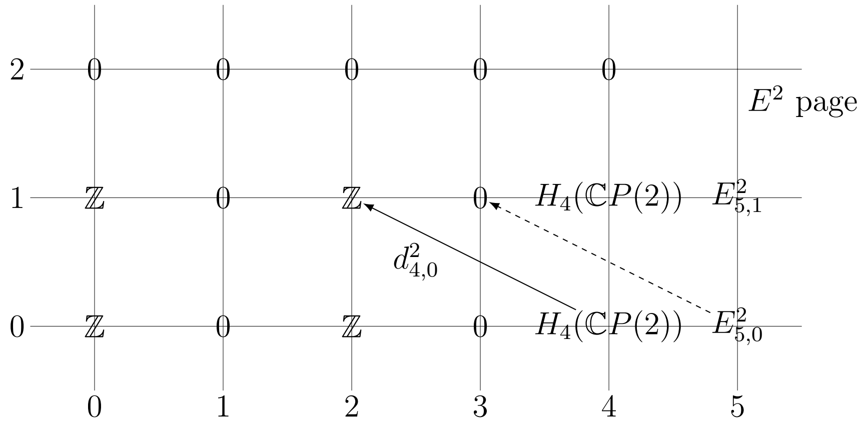
$$E^3_{0,1} = \frac{Ker(d^2_{0,1} : E^2_{0,1} \to 0)}{Im(d^2_{2,0} : E^2_{2,0} \to E^2_{0,1})} = \frac{E^2_{0,1}}{Im(d^2_{2,0})} = \frac{\mathbb{Z}}{Im(d^2_{2,0})} = 0$$

Because by the subquotient argument, $H_1(S^5;\mathbb{Z}) = 0$, so $E^3_{0,1} = E^3_{0,1} = \cdots = E^\infty_{0,1} = 0$, So $Im(d^2_{2,0}) = E^2_{0,1} = \mathbb{Z}$. $\therefore$ $d^2_{2,0}$ is surjective. Thus $d^2_{2,0}$ is an isomorphism, so $H_2(\mathbb{C}P(2);\mathbb{Z}) = \mathbb{Z}$. It follows immediately that $E^2_{2,1} = H_2(\mathbb{C}P(2);\mathbb{Z}) = \mathbb{Z}$ and this will give us our next isomorphism.

Step 3. Show $H_3(\mathbb{C}P(2);\mathbb{Z}) \cong 0$. To see this we must show $E^2_{3,0} = 0$ on the last drawn E^2 page.

$$E^3_{3,0} = \frac{Ker(d^2_{3,0} : E^2_{3,0} \to E^2_{1,1})}{Im(d^2_{5,-1} : 0 \to E^2_{3,0})} = \frac{Ker(d^2_{3,0} : E^2_{3,0} \to 0)}{0} \cong E^2_{3,0}$$

However, the abutment term $E^\infty_{3,0} = 0$ as it is a subquotient of $H_3(S^5;\mathbb{Z}) = 0$. Hence $E^2_{3,0} = E^2_{3,1} = H_3(\mathbb{C}P(2)) \cong 0$.



Step 4. Show $H_4(\mathbb{C}P(2);\mathbb{Z}) \cong \mathbb{Z}$. It suffices to show that $d^2_{4,0}$ is an isomorphism.

<u>Injectivity of $d^2_{4,0}$:</u>

$$E^3_{4,0} = \frac{Ker(d^2_{4,0} : E^2_{4,0} \to E^2_{2,1})}{Im(d_{6,-1} : E^2_{6,-1} \to E^2_{4,0})} = \frac{Ker(d^2_{4,0} : E^2_{4,0} \to \mathbb{Z})}{Im(d^2_{6,-1} : 0 \to E^2_{4,0})} = Ker(d^2_{4,0})$$

As before, $H_4(S^5;\mathbb{Z}) = 0$ so $E^\infty_{4,0}(= E^3_{4,0}) = 0$ as it is a subquotient. Thus $Ker(d^2_{4,0}) = 0$ so $d^2_{4,0}$ is an injective map.

<u>Surjectivity of $d^2_{2,0}$:</u>

$$E^3_{2,1} = \frac{Ker(d^2_{2,1} : E^2_{2,1} \to 0)}{Im(d^2_{4,0} : E^2_{4,0} \to E^2_{2,1})} = \frac{\mathbb{Z}}{Im(d^2_{4,0})}$$

However, $E^3_{2,1} = E^4_{2,1} = \cdots = E^\infty_{2,1} = 0$ as it is a subquotient of $H_3(S^5;\mathbb{Z}) = 0$. Thus, $Im(d^2_{4,0}) = E^2_{2,1} = \mathbb{Z}$ making $d^2_{4,0}$ surjective and hence an isomorphism. So $H_4(\mathbb{C}P(2);\mathbb{Z}) = \mathbb{Z}$.

Step 5. Although we know that the complex projective plane $\mathbb{C}P(2)$ has real dimension four because of its cell structure decomposition, one may wonder what about $H_5(\mathbb{C}P(2);\mathbb{Z}) = E^2_{5,0} = E^2_{5,1} = ?$. Even though the homology will be a subquotient of $H_5(S^5;\mathbb{Z}) = \mathbb{Z}$ this time, the dashed map $d^2_{5,0}$ (above) belongs to a (very) short exact sequence $0 \to H_5(\mathbb{C}P(2);\mathbb{Z}) \to 0$ and so the homology group must be trivial.

$$\underline{H_i(\mathbb{C}P(2);\mathbb{Z})} = \begin{cases} \mathbb{Z} & \text{if } i = 0,2,4 \\ 0 & \text{otherwise.} \end{cases}$$

The result can be easily generalised to compute the homology of any fibre bundle: $S^1 \to S^{2n+1} \twoheadrightarrow \mathbb{C}P(n)$ giving:

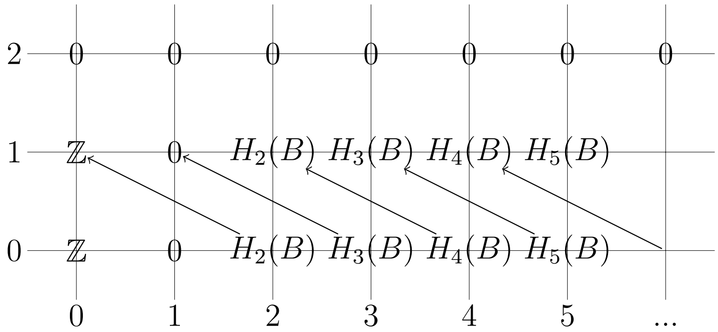
$$\underline{H_i(\mathbb{C}P(n);\mathbb{Z})} = 0 \quad \text{if } r \text{ is odd}$$

$$\underline{H_{2i}(\mathbb{C}P(n);\mathbb{Z})} = \begin{cases} \mathbb{Z} & \text{if } i \leq n \\ 0 & \text{if } i > n \end{cases}$$

Example 2. *We next consider the fibre bundle $S^1 \to S^\infty \twoheadrightarrow \mathbb{C}P(\infty)$. We know $H_i(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0,1 \\ 0 & \text{otherwise,} \end{cases}$ $H_i(S^\infty;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$*

Setting

$$E^2_{p,q} := H_p(\mathbb{C}P(\infty); H_q(S^1;\mathbb{Z})) = \begin{cases} H_p(\mathbb{C}P(\infty));\mathbb{Z} & \text{if } q = 0,1 \\ 0 & \text{otherwise.} \end{cases}$$



By the same argumentation used for $H_i(\mathbb{C}P(2);\mathbb{Z})$ we fill up the first few terms of the E^2 page above and we note that the differentials on page 2 are

$$d^2 : H_k(\mathbb{C}P(\infty)) \longrightarrow H_{k-2}(\mathbb{C}P(\infty))$$

Furthermore, these are the only differentials that have a chance of killing anything because differentials $d^r_{p,q}$ for $r \geq 3$ will travel at least two rows and start or end with a zero having trivial (co)kernels. So the E^3 page will be the same as the E^∞ page. Consider that the middle space S^∞ is contractible, so that each $E^\infty_{p,q}$ must be 0. We show that each differential is an isomorphism (except $d^2_{0,0}$) as follows:

Claim: *$H_{k+2}(\mathbb{C}P(\infty);\mathbb{Z}) \cong H_k(\mathbb{C}P(\infty);\mathbb{Z})$ for $k \geq 0$ To show this it suffices to show that:*

$$d^2_{k+2,0} : E^2_{k+2,0} \xrightarrow{\cong} E^2_{k,1}$$

is an isomorphism for all $k \geq 0$.

Injectivity of $d^2_{k+2,0}$:

$$E^3_{k+2,0} = \frac{Ker(d^2_{k+2,0} : E^2_{k+2,0} \to E^2_{k,1})}{Im(d_{k+4,-1} : 0 \to E^2_{k+2,0})} \cong Ker(d^2_{k+2,0})$$

However, since $\mathbb{C}P(\infty)$ is a subquotient of $H_i(S^\infty)$, we get $E^3_{k+2,0} = \cdots = E^\infty_{k+2,0} = 0$ so the kernel is trivial and thus $d^2_{k+2,0}$ is injective.

Surjectivity of $d^2_{k+2,0}$:

$$E^3_{k,1} = \frac{Ker(d^2_{k,1} : E^2_{k,1} \to 0)}{Im(d^2_{k+2,0} : E_{k+2,0} \to E^2_{k,1})} = \frac{E^2_{k,1}}{Im(d_{k+2,0})} = 0$$

since $E^3_{k,1} = \cdots = E^\infty_{k,1} = 0$. So $Im(d_{k+2,0}) = E^2_{k,1}$ making it surjective and thus an isomorphism. So we have shown:

$$H_{2k+1}(\mathbb{C}P(\infty);\mathbb{Z}) \cong H_{2k-1}(\mathbb{C}P(\infty);\mathbb{Z}) \cong \cdots \cong H_1(\mathbb{C}P(\infty);\mathbb{Z}) = 0$$

and

$$H_{2k}(\mathbb{C}P(\infty);\mathbb{Z}) \cong H_{2k-2}(\mathbb{C}P(\infty);\mathbb{Z}) \cong \cdots \cong H_0(\mathbb{C}P(\infty);\mathbb{Z}) = \mathbb{Z}$$

i.e.

$$\underline{H_k(\mathbb{C}P(\infty);\mathbb{Z})} = \begin{cases} \mathbb{Z} & \textit{if } k \geq 0 \textit{ is even} \\ 0 & \textit{if } k \textit{ is odd.} \end{cases}$$

We have actually computed $H_\bullet(K(\mathbb{Z},2)) = H^\bullet(K(\mathbb{Z},2))$ if we consider the fibre S^1 as a $K(\mathbb{Z},1)$ which is also a loopspace of $\mathbb{C}P(\infty)$ and the infinite sphere S^∞ as a homotopically trivial contractible space of paths in $\mathbb{C}P(\infty)$ starting at the base point, all in the setting of a fibration as the homotopy lifting property is satisfied.

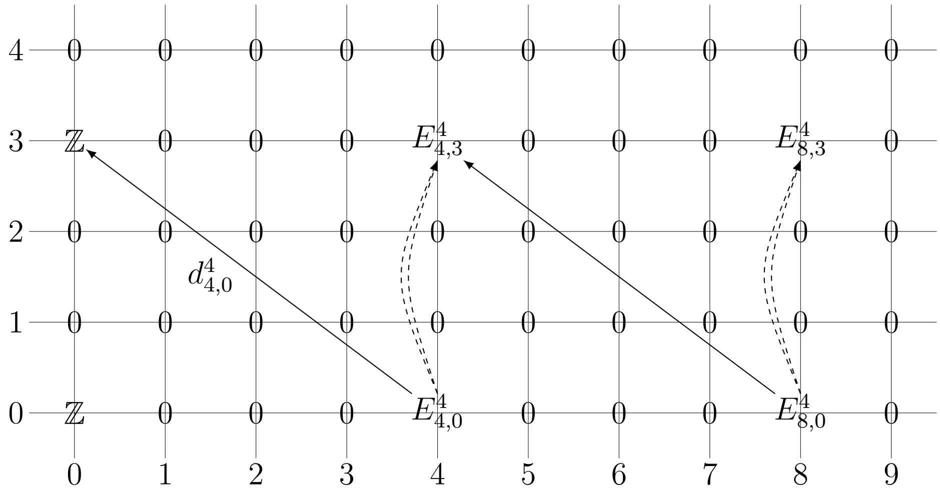
2.2. **The Quaternionic Projective Plane $\mathbb{H}P(-)$.** We use the fibre bundle $S^3 \to S^\infty \twoheadrightarrow \mathbb{H}P(\infty)$ (which can be regarded as a generalisation for the fibre bundle $S^3 \to S^{4n+3} \to \mathbb{H}P(n)$) to compute the homology of the infinite quaternionic projective space $H_*(\mathbb{H}P(\infty);\mathbb{Z})$.

We know $H_i(S^3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 3 \\ 0 & \text{otherwise,} \end{cases}$ $H_i(S^\infty;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$

Using the Leray-Serre Theorem we fill up the E^2 page with:

$$E^2_{p,q} = H_p(\mathbb{H}P(\infty); H_q(S^3;\mathbb{Z})) = \begin{cases} H_p(\mathbb{H}P(\infty);\mathbb{Z}) & \text{if } q = 0, 3 \\ 0 & \text{otherwise.} \end{cases}$$

We further assume that $\mathbb{H}P(\infty)$ is a connected space so we can fill $E^2_{0,0} = E^2_{0,3} = H_0(\mathbb{H}P(\infty);\mathbb{Z}) = \mathbb{Z}$. We can tell that $E^3_{p,q} = H(E^2_{p,q})$ will remain the same by the subquotient of $H_*(S^\infty) = 0$ argument since the differentials will have trivial (co)kernels, this only tells us that the groups stay the same from the E^2 to the E^3 pages. This leads us to consider transgressions as the only possible non-trivial differential is an edge map $d^r_{r,0} : E^r_{r,0} \to E^r_{0,r-1}$. Thus we consider the spectral sequence beginning at stage 4 (i.e. at the E^4 page):



Obviously we must study $d^4_{4,0} : E^4_{4,0} \to E^4_{0,3} = \mathbb{Z}$ which is a possible non-trivial differential. We switch attention to $E^3_{0,3} = E^2_{0,3}$ first:

$$E^4_{0,3} = \frac{Ker(d^3_{0,3} : E^3_{0,3} \to 0)}{Im(d^3_{3,1} : E^3_{3,1} \to E^3_{0,3})} = \frac{E^3_{0,3}}{0} = E^3_{0,3}$$

Now its clear that $d^4_{4,0}$ is an isomorphism since we have:

$$\begin{array}{ccc} d^4_{4,0} : E^4_{4,0} & \xrightarrow{\cong} & E^4_{0,3} \\ \Vert & & \Vert \\ E^2_{4,0} = ?\surd & & E^2_{0,3} = \mathbb{Z} \end{array}$$

$\therefore$ $E^2_{4,0} = \mathbb{Z}$ so this gives $H_4(\mathbb{H}P(\infty);\mathbb{Z}) = \mathbb{Z}$. Immediately we fill $E^4_{4,3} = \mathbb{Z}$ and extend viz the isomorphism argument used for $\mathbb{C}P(\infty)$.

We get $\underline{H_k(\mathbb{H}P(\infty);\mathbb{Z})} = \begin{cases} \mathbb{Z} & \text{if } k \equiv 0 \ mod(4), \ k \geq 0 \\ 0 & \text{otherwise.} \end{cases}$

2.3. **The Unitary Group $U(n)$.**

Definition. *We define the unitary group as a subgroup of the general linear group as follows:*

$$U(n) = \{X \in M_n(\mathbb{C}) \ : \ X \cdot \bar{X}^T = I_n\} \subset GL_n(\mathbb{C})$$

where for all complex vectors $\underline{u}, \underline{v}$ we define the inner product on $\mathbb{C}^n$ by:

$$< X(\underline{v}), X(\underline{u}) >=< \underline{u}, \underline{v} > \quad \textit{where} \quad < u, v >= \sum_{i=1}^{n} u_i \cdot \bar{v}_i$$

Now $M_n(\mathbb{C})$ act on $\mathbb{C}^n$, so $U(n)$ acts on the sphere in S^{2n-1} in $\mathbb{C}^n$:

$$U(n) \times S^{2n-1} \longrightarrow S^{2n-1}$$
$$(X \ , \ \underline{v}) \longmapsto X \cdot \underline{v}$$

where $\underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} : \sum_{i=1}^{n} |v_i|^2 = 1$ s.t. $< \underline{v}\ ,\ \underline{v} >= 1 \Leftrightarrow < X\underline{v}\ ,\ X\underline{v} >= 1$ by above. Define the standard basis $\mathcal{B} = (e_1, ..., e_n)$ and let

$$* = e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

then $U(n)_* = \{X \ : \ X \cdot * = *\} \Rightarrow X = \left(\begin{array}{c|c} X' & 0 \\ 0 & 1 \end{array} \right)$ where $X' \in U(n-1)$.

$$\therefore \quad U(n)_* \cong U(n-1)$$

so $U(n)$ acts transitively on S^{2n-1}, i.e. $\forall u \in S^{2n-1}, \exists X \in U(n)$ such that $X \cdot * = u$.

Define a map φ that gives a fibre bundle for $n \geq 2$ as follows:

$$\varphi : U(n) \longrightarrow S^{2n-1}$$
$$\text{given by} \quad \varphi(X) = X \cdot * \in U(n-1)$$
$$\Rightarrow \varphi : U(n-1) \backslash U(n) \xrightarrow{\cong} S^{2n-1}$$

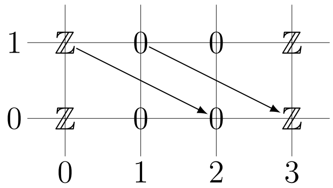
$$\begin{tikzcd} U(n-1) \arrow[r, "i"] & U(n) \arrow[d, "p"] \\ & S^{2n-1} \end{tikzcd}$$

Firstly, when $n = 1$, the group $U(1)$ corresponds to the circle group $\mathbb{S}^1 = \{z \in \mathbb{C} \ : \ |z|^2 = 1\}$ under multiplication of complex numbers. All unitary groups contain copies of this group. Also note, $dim(U(n)) = n^2$ by induction on successive fibre bundles and $H^*(S^n; \mathbb{Z}) = \overset{*}{\bigwedge}_{\mathbb{Z}}[x]$ the torsion free exterior algebra on a single generator of degree n. So $H^*(S^1; \mathbb{Z}) = \overset{*}{\bigwedge}_{\mathbb{Z}}[x_1] = \mathbb{Z} \oplus \mathbb{Z}$ generated by $\{1, x_1\}$ such that $x_1^2 = 0$.

Example 1. We attempt to compute $H_*(U(n); \mathbb{Z})$ and we realise that for $n \geq 3$ we exhaust all the various tricks used in previous examples as the d^{r+1}s cant be determined, so we need more extra structure which we obtain through cohomology spectral sequences. Note: We can recover $H_*(U(n); \mathbb{Z})$ from cohomology via the cap product. Consider the following fibre bundle which we use to compute $H^*(U(2); \mathbb{Z})$:

$$\begin{tikzcd} U(1) \arrow[r, "i"] & U(2) \arrow[d, "p"] \\ & S^3 \end{tikzcd}$$

$$E_2^{p,q} = H^p(S^3; H^q(U(1);\mathbb{Z})) = \begin{cases} H^p(S^3;\mathbb{Z}) & \text{if } q = 0,1 \\ 0 & \text{otherwise.} \end{cases} \Rightarrow H^{p+q}(U(2))$$



On the E_2 page the differentials go $d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}$ so we must have $d_2^{p,q} = 0 \ \forall p,q$.

$\therefore \ E_2^{p,q} \cong E_3^{p,q} \cong \cdots \cong E_\infty^{p,q}$ so the groups above stabilize to the E_∞ page. We use the Künneth Theorem for cohomology:

$$\begin{aligned} H^n(U(2);\mathbb{Z}) &\cong \bigoplus_{p+q=n} H^p(S^3; H^q(S^1;\mathbb{Z})) \\ &\cong \bigoplus_{p+q=n} H^p(S^3;\mathbb{Z}) \otimes H^q(S^1;\mathbb{Z}) \end{aligned}$$

$$\therefore \ \underline{H^k(U(2);\mathbb{Z})} \cong H^*(S^3;\mathbb{Z}) \otimes H^*(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0,1,3,4 \\ 0 & \text{if } k = 2 \text{ or } k \geq 5. \end{cases}$$

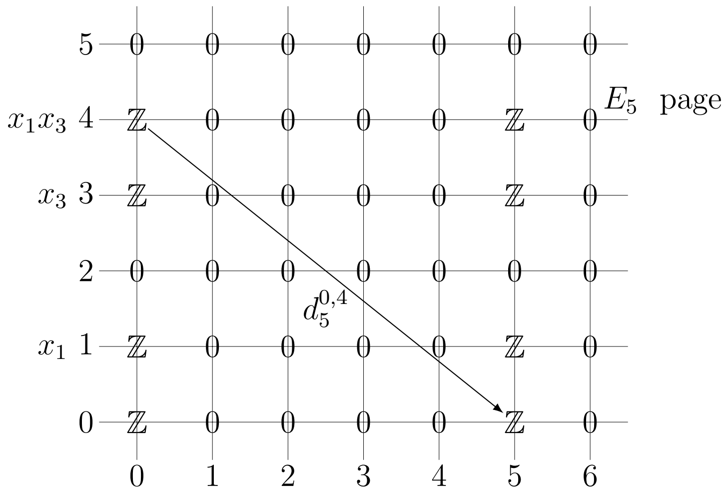
Hence we see that $U(2)$ behaves from the cohomology point of view as if we've got $S^3 \times S^1$, infact $U(2) \cong S^3 \times S^1$. Since $H^*(S^3;\mathbb{Z}) = \overset{*}{\bigwedge}[x_3]$ we get $H^*(U(2);\mathbb{Z}) = \overset{*}{\bigwedge}[x_1, x_3]$ such that we have $x_1^2 = x_3^2 = 0$ and $x_1 \wedge x_3 = -x_3 \wedge x_1$. Infact, $H^*(S^p \times S^q) = \bigwedge[x,y]$ s.t. $x \in H^p$ and $y \in H^q$.

Example 2. Using cohomology spectral sequences, consider the fibre bundle:

$$\begin{tikzcd} U(2) \arrow[r, "i"] & U(3) \arrow[d, "p"] \\ & S^5 \end{tikzcd}$$

$$E_2^{p,q} = H^p(S^5; H^q(U(2);\mathbb{Z})) = \begin{cases} H^p(S^5;\mathbb{Z}) & q = 0,1,3,4 \\ 0 & \text{otherwise.} \end{cases}$$
$$\Rightarrow H^{p+q}(U(3);\mathbb{Z})$$

Since $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ we only get one intersting transgressional differential on the E_5 page which may be non-trivial, namely $d_5^{0,4} : E_5^{0,4} \to E_5^{5,0}$ which is an edge homomorphism. We use the Leibintz rule which the differential satisfies on the exterior algebra generators to show $d_5^{0,4} = 0$:



$$\begin{aligned} d(x_1 \wedge x_3) &= d(x_1) \wedge x_3 \pm x_1 \wedge d(x_3) \\ &= 0 \wedge x_3 \pm x_1 \wedge 0 \\ &= 0. \end{aligned}$$

So we must have $E_2^{p,q} = E_3^{p,q} = \cdots = E_\infty^{p,q}$.

$$\begin{aligned} \therefore\ \underline{H^*(U(3);\mathbb{Z})} &= H^*(S^1;\mathbb{Z}) \otimes H^*(S^3;\mathbb{Z}) \otimes H^*(S^5;\mathbb{Z}) \\ \text{“reading off”} &= \bigoplus_{p+q=n} E_\infty^{p,q} = \bigwedge^*[x_1, x_3, x_5] \end{aligned}$$

such that $x_1^2 = x_3^2 = x_5^2 = 0,\ x_i \wedge x_j = -x_j \wedge x_i\ \forall i < j$.

Theorem. *We generalise our results and prove the following by induction on n:*

$$\begin{aligned} H^*(U(n);\mathbb{Z}) &= H^*(S^1 \times S^3 \times \cdots \times S^{2n-1};\mathbb{Z}) \\ &= H^*(S^1;\mathbb{Z}) \otimes H^*(S^3;\mathbb{Z}) \otimes \cdots \otimes H^*(S^{2n-1};\mathbb{Z}) \\ &= \bigwedge\nolimits_{\mathbb{Z}}^*[x_1, ..., x_n] \quad \textit{where} \quad x_i \in H^{2i-1}(U(n);\mathbb{Z}). \end{aligned}$$

Proof. Induction base is done for $U(2)$, for the induction hypothesis we assume:

$$H^*(U(n-1);\mathbb{Z}) \cong H^*(S^1 \times S^3 \times \cdots \times S^{2n-3};\mathbb{Z}) \cong \bigwedge^*[x_1, x_3, ..., x_{2n-3}]$$

Take the fibre bundle $U(n-1) \hookrightarrow U(n) \twoheadrightarrow S^{2n-1}$ and use Serre-Leray:

$$E_2^{p,q} = H^p(S^{2n-1}; H^q(U(n-1);\mathbb{Z})) \Rightarrow \bigoplus_{p+q=n} E_\infty^{p,q} = H^{p+q}(U(n);\mathbb{Z})$$

since the only possible non-zero differential which occurs on the E_{2n-1} page is $d_{2n-1}^{0,2n-2} : E_{2n-1}^{0,2n-2} \to E_{2n-1}^{2n-1,0}$ and since

$$d(x_1 \wedge x_{2n-3}) = d(x_1) \wedge x_{2n-3} \pm x_1 \wedge d(x_{2n-3}) = 0 \pm 0 = 0$$

Whence the result $\underline{H^*(U(n);\mathbb{Z})} = \bigwedge^*[x_1, ..., x_n]$ with $x_i \in H^{2i-1}(U(n))$.[1]

$\square$

2.4. **Eilenberg-MacLane Spaces $K(G, n)$.** We essentially computed the cohomology of $K(\mathbb{Z}, 2)$ by computing $H^\bullet(\mathbb{C}P(\infty);\mathbb{Z})$. So we next try to find: $H^*(K(\mathbb{Z}, 3))$ by considering the pathspace fibration $F \to S^\infty \to B$, where B is a $K(\mathbb{Z}, 3)$, and since S^∞ is contractible we use the long exact sequence of a fibration

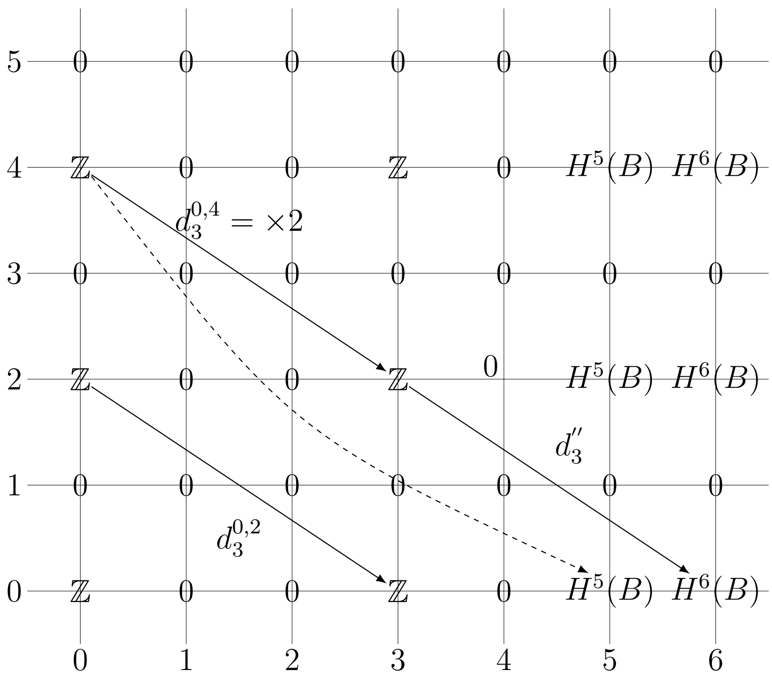
$$\cdots \to \pi_i(F) \to \pi_i(S^\infty) \to \pi_i(B) \to \pi_{i-1}(F) \to \pi_{i-1}(S^\infty) \to \cdots$$

for $i > 0$ to show $\pi_i(B) \cong \pi_{i-1}(F)$ so $\pi_i(K(\mathbb{Z}, 3)) \cong \pi_{i-1}(K(\mathbb{Z}, 2))$ hence F is a $K(\mathbb{Z}, 2)$ (and so homotopy equivalent to $\mathbb{C}P(\infty)$). By the Hurewicz Theorem, we know that $H^3(K(\mathbb{Z}, 3)) = \mathbb{Z}$ and by taking a model of $K(\mathbb{Z}, 3)$ with no 4-cells such that $H^4(K(\mathbb{Z}, 3)) = H_4(K(\mathbb{Z}, 3)) = 0$ which gives us more leverage, we can use the U.C.T.C. to see that:

$$H^5(K(\mathbb{Z}, 3)) = Ext(H_4(K(\mathbb{Z}, 3), \mathbb{Z})) \oplus Hom(H_5(K(\mathbb{Z}, 3)), \mathbb{Z}) = 0$$

Using Leray-Serre we fill up the E_2 page which has infinitely many non-trivial even rows:

$$E_2^{p,q} = H^p(K(\mathbb{Z}, 3); H^q(\mathbb{C}P(\infty);\mathbb{Z}))$$



[1]$H^*(\mathbb{C}P(n);\mathbb{Z}) = \mathbb{Z}[y]/(y^{n+1})$ is a polynomial algebra with one generator s.t. $deg(y) = 2$ and $H^*(\mathbb{H}P(n);\mathbb{Z}) = \mathbb{Z}[z]/(z^{n+1})$ is a stunted polynomial ring with a single generator z s.t. $deg(z) = 4$.

So lets try to calculate to $H^5(K(\mathbb{Z},3))$ and $H^6(K(\mathbb{Z},3))$. Since every other row is 0, all the differentials on the E_2 page are trivial, so $E_3^{p,q} = E_2^{p,q}$. Using the Leibintz rule on the differentials we let $x \in H^0(K(\mathbb{Z},3))$ and $y \in H^3(K(\mathbb{Z},3))$ be generators, then consider the map on the E_3 page:

$$d_3 : E_3^{0,2} \to E_3^{3,0}$$
$$d_3 : H^0(K(\mathbb{Z},3); H^2(\mathbb{C}P(\infty))) \to H^3(K(\mathbb{Z},3); H^0(\mathbb{C}P(\infty)))$$

We need to understand the image of $x = x \cdot 1$. Upon inspection we see that the only differential which can affect the terms $E_\infty^{0,2}$ or $E_\infty^{3,0}$ is the differential $d_3^{0,2}$ on page 3 above. Furthermore, we know that $H^i(S^\infty) = 0$ for all $i > 0$, and so $E_\infty^{p,q} = 0$ whenever either p or q is non-zero. Thus the $\mathbb{Z}$s on $E_3^{0,2}$ and $E_3^{3,0}$ will get killed so $E_\infty^{0,2}$ and $E_\infty^{3,0}$ are both zero. This can only happen if $d_3(x)$ is a generator for $H^3(K(\mathbb{Z},3))$ and so we choose the sign of y so that $d_3(x) = y$. Similarly, we can move up and use the multiplicative structure to calculate:

$$d_3' : E_3^{0,4} \to E_3^{3,2}$$
$$d_3' : H^0(K(\mathbb{Z},3); H^4(\mathbb{C}P(\infty))) \to H^3(K(\mathbb{Z},3); H^2(\mathbb{C}P(\infty)))$$

This time the generator for the left term is x^2, and we "break it down" viz $d_3'(x^2) = d_3'(x)x + xd_3'(x) = y \cdot x + x \cdot y$ and since x has even degree, the cup product of x and y is commutative, so we get $y \cdot x + x \cdot y = 2x \cdot y$. In particular, this tells us that the differential d_3' is a times 2 map. We deduce that on the next page $E_4^{0,4}$ is zero since the differential is injective.

We use the dashed map on the "E_5 page" to show that $E_3^{5,0} = H^5(K(\mathbb{Z},3)) = 0$ since the only differential with a chance to kill it is:

$$d_5'' : E_5^{0,4} \to E_5^{5,0} = H^5(K(\mathbb{Z},3))$$

and so the differential must be zero.

Furthermore, the differential d_3'' on the E_3 page:

$$\mathbb{Z}/2\mathbb{Z} = E_3^{3,2}/Im(d_3') \xrightarrow{d_3''} E_3^{6,0} = H^6(K(\mathbb{Z},3))$$

must be an isomorphism; indeed everything on the $p+q=5$ and $p+q=6$ diagonals must eventually die, and this is the last differential with a chance to kill either. We conclude that:

$$H^6(K(\mathbb{Z},3);\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

Over rational coefficients, we get a more general result by induction:

$$\underline{H^\bullet(K(\mathbb{Z},n);\mathbb{Q})} = \begin{cases} \mathbb{Q}[x] & x \in H^n(K(\mathbb{Z},n),\ n \text{ even} \\ \mathbb{Q}[x]/x^2 & x \in H^n(K(\mathbb{Z},n),\ n \text{ odd}. \end{cases}$$